Part IA
Dynamics and Relativity

Prof Sean Hartnoll

Lent 2026
Version 20260609

These are Zixuan's notes for **Part IA – Dynamics and Relativity** at the University of Cambridge in 2026. The notes are not endorsed by the lecturers or the University, and all errors are my own.

The latest version of this document is available at academic.micfong.space. Please direct any comments to my CRSid email or use the contact details listed on the site.

This document is typeset using Typst. All figures are created using Inkscape.

Contents

Syllabus and Overview

Lent Term, 2026 [24 Lectures]

Faded topics are not examinable.

Basic Concepts [4 Lectures]

Space and time, frames of reference, Galilean transformations. Newton's laws. Dimensional analysis. Examples of forces, including gravity, friction and Lorentz.

Newtonian Dynamics of a Single Particle [8 Lectures]

Equation of motion in Cartesian and plane polar coordinates. Work, conservative forces and potential energy, motion and the shape of the potential energy function; stable equilibria and small oscillations; effect of damping.

Angular velocity, angular momentum, torque.

Orbits: the $u(\theta)$ equation; escape velocity; Kepler's laws; stability of orbits; motion in a repulsive potential (Rutherford scattering).

Rotating frames: centrifugal and Coriolis forces. Brief discussion of Foucault pendulum.

Newtonian Dynamics of Systems of Particles [2 Lectures]

Momentum, angular momentum, energy. Motion relative to the centre of mass; the two body problem. Variable mass problems; the rocket equation.

Rigid Bodies [3 Lectures]

Moments of inertia, angular momentum and energy of a rigid body. Parallel axis theorem. Simple examples of motion involving both rotation and translation (e.g. rolling).

Special Relativity [7 Lectures]

The principle of relativity. Relativity and simultaneity. The invariant interval. Lorentz transformations in $(1+1)$-dimensional spacetime. Time dilation and length contraction. The Minkowski metric for $(1+1)$-dimensional spacetime.

Lorentz transformations in $(3+1)$ dimensions. 4–vectors and Lorentz invariants. Proper time. 4–velocity and 4–momentum. Conservation of 4–momentum in particle decay. Collisions. The Newtonian limit.

Lecture 1 · 2026-01-22

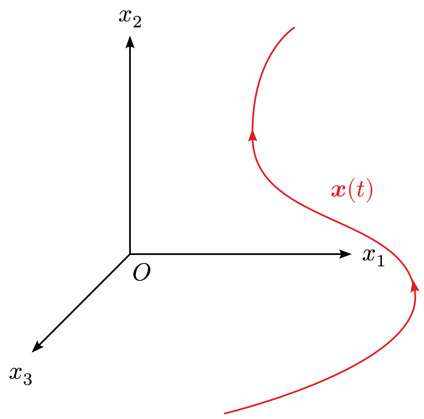
1 The Structure of the Newtonian Universe

To set up the arena we are going we work in, we require

- a three dimensional **space** that can be endowed with a *Cartesian reference frame* [*i.e.* an origin and some axes], such that points in space are labelled as

$$\boldsymbol{x} = (x_1, x_2, x_3). \quad \cdot 1$$

- a **time** parameter that can be labelled, in an arbitrary reference frame, by a real number t.
- a **point particle** which is an idealised object that is completely determined by its position at a given time as $\boldsymbol{x}(t)$.

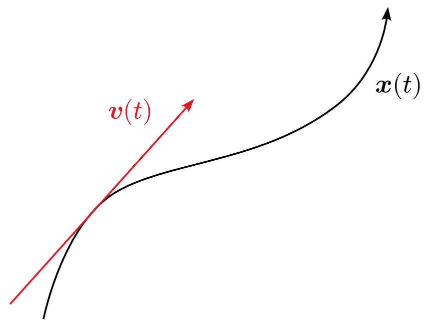


Examples include electron, tennis ball, planet depending on the context.

- the **velocity** which is the vector

$$\boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \dot{\boldsymbol{x}}. \quad \cdot 2$$

From results in IA Vector Calculus, the velocity vector is tangent to the trajectory of the particle.



Recall that in Cartesian coordinates,

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \left(\frac{\mathrm{d}x_1}{\mathrm{d}t}, \frac{\mathrm{d}x_2}{\mathrm{d}t}, \frac{\mathrm{d}x_3}{\mathrm{d}t}\right). \quad \cdot 3$$

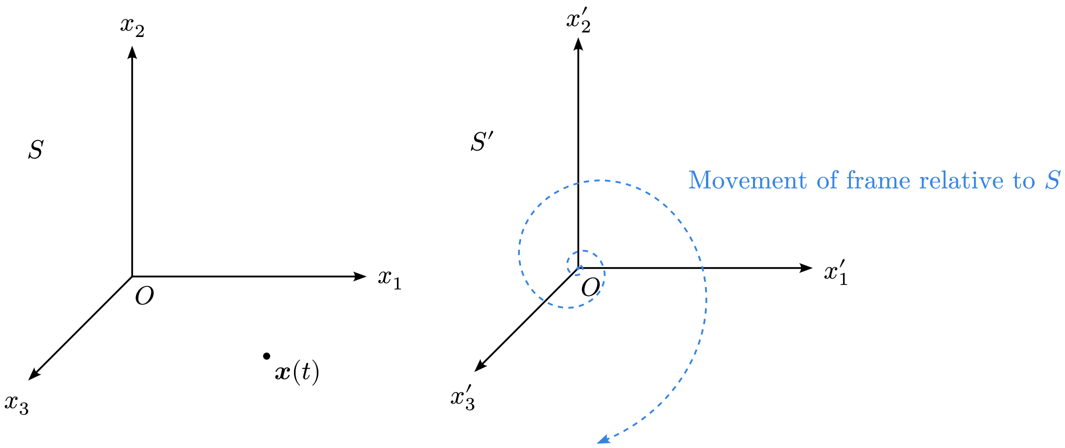
[We will discuss other coordinate systems later in the course.]

- the **acceleration** which is the vector

$$\boldsymbol{a} = \ddot{\mathbf{x}} = \dot{\mathbf{v}} = \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}. \tag{4}$$

The above structure is not enough to write down Newton's equations.

Consider a "free" particle that does not experience any forces. [*e.g.* the particle is alone in deep space far away from any other matter] The position of this particle is $\boldsymbol{x}(t)$, we need to consider which reference frame we are using.



The particle may be at rest in a frame S, but moving in a complicated way with respect to another frame S'.

Law 1.1 (Law of Inertia)

There exists **inertial frames**, in which a free particle has a constant velocity.

In an inertial frame, we may write for a free particle,

$$\dot{\mathbf{v}} = \ddot{\mathbf{x}} = \mathbf{0}. \tag{5}$$

The law of inertia is an improved version of Newton's 1st law.

This is a true statement about the world, but not an obvious one. [In antiquity, it was believe that the natural state of an object is to be at rest, and a force is required to keep it moving.]

Law 1.2 (Galilean Relativity Principle)

A frame related to an inertial frame by a **Galilean transformation** is also an inertial frame, and all laws of physics are the same in both frames.

Definition 1.3 (Galilean Transformation)

A **Galilean transformation** between two reference frames S and S' is given by

$$\boldsymbol{x}' = \mathbf{R}\mathbf{x} + \boldsymbol{k} + \boldsymbol{w}t \tag{6}$$

where $\mathbf{R} \in O(3)$ is a rotation and/or a reflection, $\boldsymbol{k} \in \mathbb{R}^3$ is a constant translation, and $\boldsymbol{w} \in \mathbb{R}^3$ is a constant velocity, called a boost.

It is easy to see that $\ddot{\mathbf{x}} = \mathbf{0} \Leftrightarrow \ddot{\mathbf{x}}' = \mathbf{0}$ under a Galilean transformation.

Example 1.4

Consider a frame relative to a boat moving at constant velocity. A mass dropped from the mast of the boat lands at the same place on this boat as if the boat were not moving.

Galilean invariance restricts the type of forces that are possible (see Example Sheet 1).

Galilean relativity implies that the laws of physics make reference to no special point, direction, time or velocity. All these things can only be defined relatively. [*e.g.* one cannot be "at rest", but only at rest with respect to something.]

Acceleration is not relative. If one is accelerating in an inertial frame, then they will be accelerating in all other inertial frames, with the same magnitude.

Remark. The Galilean transformations form the **Galilean group**, often supplemented with time translations:

$$t' = t + t_0. \quad \cdot 7$$

Laws of physics are also invariant under time translations, *i.e.* all inertial frames have the same time, called **absolute time**.

2 Forces

Once there is more than one particle in the universe, there will be interactions between the particles. In Newtonian physics, these are described by **forces**.

2.1 Newton's Second Law

Law 2.1 (Newton's 2nd Law)

In an inertial frame,

$$\dot{\boldsymbol{p}} = \boldsymbol{F}, \tag{8}$$

where $\boldsymbol{p}$ is the momentum of the particle and $\boldsymbol{F}$ is the net force acting on the particle.

The **momentum** $\boldsymbol{p}$ is defined to be $\boldsymbol{p} \equiv m\dot{\boldsymbol{x}}$, where m is the inertial mass.

The mass is an additional property of particles. It could change with time, but we generally assume it is constant unless otherwise specified.

The force $\boldsymbol{F}$ depends on the interaction, but can only depend on $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ at the current time.

The above implies that

- Newton's second law can be written as a second order ODE for $\boldsymbol{x}(t)$.
- given $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ at $t = 0$ for all particles, Newton's equations uniquely determine $\boldsymbol{x}(t)$ for all future times.

Important. Newton mechanics has been superceded by both quantum mechanics (small scale) and relativity (high speed), but remains an excellent approximation much of the universe.

Lecture 2 · 2026-01-24

2.2 Conservative Forces and Gravity

Definition 2.2 (Conservative Force)

Conservative forces form an important class of forces that can be written as

$$\boldsymbol{F} = -\nabla V(\boldsymbol{x}) \tag{9}$$

for some **potential** (also called **potential energy**) V.

Remark. Recall from IA Vector Calculus that $\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$.

Example 2.3 (Gravitational Force)

The gravitational potential energy of a particle of mass m at $\boldsymbol{x}$ due to a particle of mass M at $\boldsymbol{x}_0$ is

$$V = -\frac{GMm}{|\mathbf{x} - \mathbf{x}_0|} \quad \cdot 10$$

where $G \approx 6.67 \cdot 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$.

To take the gradient,

$$\partial_i\left(|\mathbf{x} - \mathbf{x}_0|^2\right) = 2\,|\mathbf{x} - \mathbf{x}_0|\,\partial_i\,|\mathbf{x} - \mathbf{x}_0| \quad \cdot 11$$

and

$$\begin{aligned}
\partial_i\left(|\mathbf{x} - \mathbf{x}_0|^2\right) &= \partial_i\left((\mathbf{x} - \mathbf{x}_0)_j(\mathbf{x} - \mathbf{x}_0)_j\right) && \cdot 12 \\
&= 2\left((\mathbf{x} - \mathbf{x}_0)_j\partial_i(\mathbf{x} - \mathbf{x}_0)_j\right) && \cdot 13 \\
&= 2\left((\mathbf{x} - \mathbf{x}_0)_j\delta_{ij}\right) && \cdot 14 \\
&= 2\left((\mathbf{x} - \mathbf{x}_0)_i\right). && \cdot 15
\end{aligned}$$

Hence

$$\nabla\,|\mathbf{x} - \mathbf{x}_0| = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}. \quad \cdot 16$$

This gives

$$\mathbf{F} = -\nabla V = -\frac{GMm}{|\mathbf{x} - \mathbf{x}_0|^3}(\mathbf{x} - \mathbf{x}_0). \quad \cdot 17$$

If we let $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$, this is the familiar inverse square law

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}. \quad \cdot 18$$

Sometimes we write $V = m\Phi$, where Φ is the gravitational potential

$$\Phi = -\frac{GM}{|\mathbf{x} - \mathbf{x}_0|}. \quad \cdot 19$$

Near the surface of the Earth, take $\mathbf{x}_0 = \mathbf{0}$ the centre of the Earth, and $|\mathbf{x}| = R + z$, where R is the radius of the Earth and $z \ll R$ is the height above the surface. Then

$$\begin{aligned}
\Phi(R + z) &= -\frac{GM}{R + z} && \cdot 20 \\
&\approx -\frac{GM}{R}\left[1 - \frac{z}{R} + \frac{z^2}{R^2} - \dots\right] && \cdot 21 \\
&\approx \text{constant} + \underbrace{\frac{GM}{R^2}}_{g \approx 9.8\,\mathrm{m\,s^{-2}}} z + \dots && \cdot 22
\end{aligned}$$

Thus, near the surface of the Earth, we approximate the gravitational potential as

$$\Phi(z) \approx \underbrace{\text{constant}}_{\text{drops out at gradient}} + gz. \tag{23}$$

The force is then

$$\boldsymbol{F} = -mg\hat{\boldsymbol{z}}, \tag{24}$$

which is a constant force near the surface of the Earth.

This force leads to the simplest example of motion due to a force.

Newton's 2nd law gives

$$m\ddot{\boldsymbol{x}} = m\boldsymbol{g} \tag{25}$$

where $\boldsymbol{g} = (0, 0, -g)$. Consider the z-component,

$$\cancel{m}\ddot{z} = -\cancel{m}g \tag{26}$$

which gives

$$\dot{z} = v_0 - gt \tag{27}$$

$$z = z_0 + v_0 t - \frac{1}{2}gt^2. \tag{28}$$

where z_0 and v_0 are the initial position and velocity at $t = 0$.

2.3 Conservation of Energy

Proposition 2.4 (Conserved Energy of Conservative Forces)

Conservative forces have a **conserved energy**

$$E = \frac{1}{2}m|\dot{\boldsymbol{x}}|^2 + V(\boldsymbol{x}). \tag{29}$$

We can check that this is conserved:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = m\dot{x}_i\ddot{x}_i + \frac{\partial V}{\partial x_i}\dot{x}_i \tag{30}$$

$$= \dot{x}_i\left(m\ddot{x}_i + \frac{\partial V}{\partial x_i}\right) \tag{31}$$

$$= 0. \quad \text{by Newton } m\ddot{\boldsymbol{x}} = -\boldsymbol{\nabla}V \tag{32}$$

Example 2.5

Suppose we throw an object into space and want it to never fall back down. The minimal velocity this object must have is called the **escape velocity**.

As the object is thrown,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}. \quad \cdot 33$$

To not fall back, the object must be able to reach $|x| \to \infty$ without the velocity going to zero. Hence

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = E_{\text{initial}} \quad \cdot 34$$
$$= E_\infty \quad \cdot 35$$
$$= \frac{1}{2}mv_0^2 - 0 > 0. \quad \cdot 36$$

Therefore we require

$$v^2 > \frac{2GM}{R}. \quad \cdot 37$$

Hence the escape velocity is

$$v_{\text{escape}} = \sqrt{\frac{2GM}{R}} \approx 10\,\text{km}\,\text{s}^{-1} \text{ for the Earth.} \quad \cdot 38$$

The mass m is cancelled out, since the gravitational mass (that appears in the inverse square law) is the same as the inertial mass (that appears in Newton's 2nd law).

It is useful to write $E = T + V$, where $T = \frac{1}{2}m|\dot{\mathbf{x}}|^2$ is the **kinetic energy** and V is the **potential energy**.

Proposition 2.6

Conservative forces have the property that the work done by the force as a particle moves along a trajectory C, where the work done is defined as

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}, \quad \cdot 39$$

only depends on the endpoints of the trajectory, not on the path itself.

Proof. Let the trajectory C go from $\mathbf{x}_1$ at t_1 to $\mathbf{x}_2$ at t_2. Then

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} \quad \cdot 40$$
$$= \int_{t_1}^{t_2} \underbrace{\mathbf{F} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}}_{\text{power}} \mathrm{d}t \quad \cdot 41$$
$$= m \int_{t_1}^{t_2} \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}\, \mathrm{d}t \qquad \text{(Newton's 2nd law)} \quad \cdot 42$$
$$= \frac{1}{2}m \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(|\dot{\mathbf{x}}|^2\right) \mathrm{d}t \quad \cdot 43$$
$$= T(t_2) - T(t_1) \quad \cdot 44$$

$$= V(t_1) - V(t_2) \qquad \text{(Conservation of energy)} \tag{45}$$
$$= V(\boldsymbol{x}(t_1)) - V(\boldsymbol{x}(t_2)) \tag{46}$$
$$= V(\boldsymbol{x}_1) - V(\boldsymbol{x}_2). \tag{47}$$

Lecture 3 · 2026-01-27

Proof (Direct). Using results from IA Vector Calculus, we have

$$W = \int_C \boldsymbol{F}\, \mathrm{d}\boldsymbol{x} \tag{48}$$
$$= -\int_C \boldsymbol{\nabla} V\, \mathrm{d}\boldsymbol{x} \tag{49}$$
$$= -\int_{x_1}^{x_2} dV \tag{50}$$
$$= V(\boldsymbol{x}_1) - V(\boldsymbol{x}_2). \tag{51}$$

2.4 Electromagnetic Forces

Forces that depend on the velocity typically don't have a conserved energy, such as friction. However, the **Lorentz force** is an exception.

Electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ exert the following force on a particle with charge q

$$\boldsymbol{F} = q[\boldsymbol{E}(\boldsymbol{x}) + \dot{\boldsymbol{x}} \times \boldsymbol{B}(\boldsymbol{x})]. \tag{52}$$

In this section, we shall restrict to static electromagnetic fields, i.e., $\boldsymbol{E}$ and $\boldsymbol{B}$ do not depend on time. Then

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi, \tag{53}$$

where $\Phi(\boldsymbol{x})$ is the electric potential.

We claim that the conserved energy is

$$E = \frac{1}{2}m|\dot{\boldsymbol{x}}|^2 + q\Phi(\boldsymbol{x}). \tag{54}$$

To check this,

$$\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} = m\ddot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} + q\boldsymbol{\nabla}\Phi \cdot \dot{\boldsymbol{x}} \tag{55}$$
$$= (\boldsymbol{F} + q\boldsymbol{\nabla}\Phi) \cdot \dot{\boldsymbol{x}} \tag{56}$$
$$= q(\dot{\boldsymbol{x}} \times \boldsymbol{B}) \cdot \dot{\boldsymbol{x}} \qquad \text{(by Lorentz)} \tag{57}$$
$$= 0. \tag{58}$$

The velocity-dependent force is orthogonal to the trajectory of the particle, so it does no work.

Electric forces are similar to gravitational ones. The potential Φ at $\boldsymbol{x}$ due to another particle of charge Q at $\boldsymbol{x}_0$ is

$$\Phi = \frac{Q}{4\pi\varepsilon_0} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|}, \tag{59}$$

where ε_0 is the permittivity of free space, approximately $8.85 \cdot 10^{-12}\,\mathrm{m}^{-3}\,\mathrm{kg}^{-1}\,\mathrm{s}^2\,\mathrm{C}^2$.

Like gravity, this leads to an inverse square law for the electric force, called **Coulomb's law**. However, charges can be positive or negative, but mass is always positive. Hence, gravity dominates for large objects while electric foces tend to cancel out overall.

For magnetic forces, a charged particle in a magnetic field $\boldsymbol{B}$ obeys

$$m\ddot{\boldsymbol{x}} = q\dot{\boldsymbol{x}} \times \boldsymbol{B}. \tag{60}$$

This is a vector differential equation. The most direct way to solve it is to write out components.

Suppose $\boldsymbol{B}$ is constant and WLOG along the z-axis, i.e., $\boldsymbol{B} = (0, 0, B) = B\hat{\boldsymbol{z}}$. The equations become

$$\begin{cases} (1) \quad m\ddot{x} = qB\dot{y} \\ (2) \quad m\ddot{y} = -qB\dot{x} \\ (3) \quad m\ddot{z} = 0 \qquad \Rightarrow z = z_0 + v_z t. \end{cases} \tag{61}$$

Solution 1. Using $\frac{\mathrm{d}}{\mathrm{d}t}$ (1) and (2),

$$m\dddot{x} = qB\ddot{y} = -\frac{q^2\dot{x}B^2}{m} \tag{62}$$

which is a 2nd order equation for $\dot{x}$. This gives

$$\dot{x} = \tilde{A}\sin(\omega t + \varphi) \tag{63}$$

where

$$\omega = \frac{qB}{m} \tag{64}$$

which is called the **cyclotron frequency**.

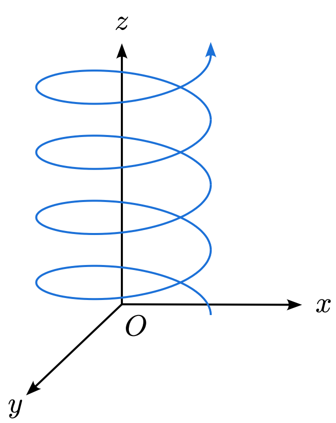
This gives

$$x = x_0 + A\cos(\omega t + \varphi). \tag{65}$$

Substituting back into (1) gives

$$qB\dot{y} = -mA\omega^2\cos(\omega t + \varphi) \tag{66}$$
$$y = y_0 - A\sin(\omega t + \varphi) \tag{67}$$

where A and φ are arbitrary.



The period T is the time to complete one cycle,

$$T = \frac{2\pi}{\omega}. \tag{68}$$

Solution 2. Alternatively, let $\xi = x + \mathrm{i}\, y$. Then we can write (1) + i (2) as

$$m\ddot{\xi} = -\mathrm{i}\, qB\dot{\xi}. \tag{69}$$

Solving gives

$$\xi = C_1 \mathrm{e}^{-\mathrm{i}\,\omega t} + C_2 \tag{70}$$

where C_1 and C_2 are complex constants.

Set $C_1 = A\mathrm{e}^{-\mathrm{i}\,\varphi}$ and $C_2 = x_0 + \mathrm{i}\, y_0$. Then taking real and imaginary parts recovers

$$\begin{cases} x = x_0 + A\cos(\omega t + \varphi) \\ y = y_0 + A\sin(\omega t + \varphi). \end{cases} \tag{71}$$

Remark. There is something "complex" underlying cyclotron motion, *c.f.* quantum hall effect.

2.5 Motion in One Dimension

Problems can often be reduced to one-dimensional motion, such as

$$m\ddot{x} = F_x. \tag{72}$$

If F_x is independent of velocity, then it can always be written in terms of a potential, by setting

$$V(x) = -\int_{x_0}^{x} \mathrm{d}x' F_x(x') \tag{73}$$

where x_0 is an arbitrary reference point. This gives

$$F_x(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{74}$$

The following energy is then conserved:

$$E = \frac{1}{2}m\dot{x}^2 + V(x). \tag{75}$$

Keeping E constant gives a 1st order ODE for $x(t)$, which is easy to integrate:

$$\dot{x} = \pm\sqrt{\frac{2}{m}(E - V(x))} \tag{76}$$

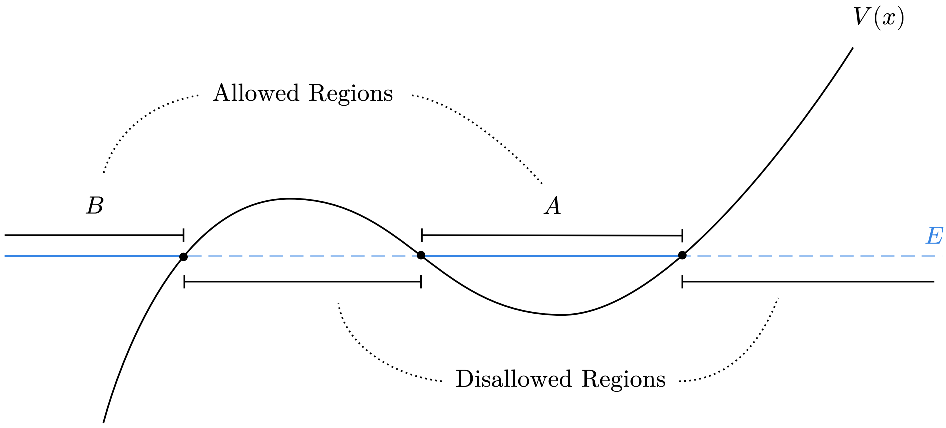
$$t - t_0 = \pm\int_{x_0}^{x} \frac{\mathrm{d}x'}{\sqrt{\frac{2}{m}(E - V(x'))}}. \tag{77}$$

This equation tells us how long it takes to move from x_0 to x, if it has energy E.

More often than not the integral is not analytically solvable, but it is still useful for qualitative analysis of motion.

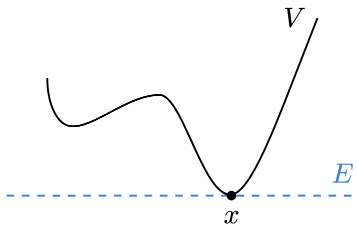
Lecture 4 · 2026-01-29

From $E = \frac{1}{2}m\dot{x}^2 + V(x)$, we get $E > V(x)$. This restricts the range of x where the particle can be.



The points where $E = V(x) \Rightarrow \dot{x} = 0$ are called turning points. Typically, particles bounces off the potential at turning points and turn around.

- Within region A, the particle bounces back and forth between turning points (bounded motion).
- Within region B, the particle bounces off the turning point and escapes to $x \to -\infty$ (unbounded motion).
- A special case occurs when $E = V(x)$ and $V'(x) = 0$. These are equilibrium points, where the particle can remain at rest if placed there.

  We have $m\ddot{x} = -V'(x) = 0$ and $E = \frac{1}{2}m\dot{x}^2 + V(x)$, giving $\dot{x} = \ddot{x} = 0$.

We shall show that motion close to equilibrium points are especially simple.

Let x_0 be the equilibrium point, the Taylor expansion about x_0 gives

$$\begin{aligned} V(x) &\approx V(x_0) + (x - x_0)V'(x) + \frac{1}{2}(x - x_0)^2 V''(x_0) + \dots && \cdot 78 \\ &\approx V(x_0) + \frac{1}{2}(x - x_0)^2 V''(x_0) && \text{since } V'(x_0) = 0. \quad \cdot 79 \end{aligned}$$

- If $V''(x_0) > 0$, this is the potential for a simple harmonic oscillator,

  $$m\ddot{x} = -V'(x) \approx -(x - x_0)V''(x_0). \tag*{· 80}$$

  Solving this gives

  $$x = x_0 + A\cos(\omega t + \varphi) \tag*{· 81}$$

  where A is the amplitude and φ is the phase. The angular frequency is

$$\omega = \sqrt{\frac{V''(x_0)}{m}}. \quad \cdot 82$$

If A is small enough, we can neglect higher order terms in the Taylor expansion, and the oscillating solution is valid. The point x_0 is called a **stable equilibrium**.

- If $V''(x_0) < 0$, then we get an **unstable equilibrium** point. The solution is

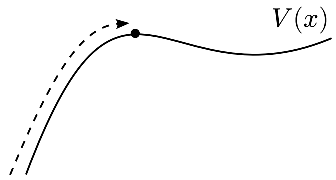
$$x = x_0 = \tilde{A}e^{\gamma t} + \tilde{B}e^{-\gamma t} \quad \cdot 83$$

where

$$\gamma = \sqrt{\frac{-V''(x_0)}{m}}. \quad \cdot 84$$

If $\tilde{A} \neq 0$, the exponential growth means the particle moves far away from the equilibrium point, so the approximation breaks down.

The case $\tilde{A} = 0$ corresponds to rolling the particle up the potential with just enough energy to reach the top as $t \to \infty$.



If $V''(x_0) = 0$, we need to include higher order terms in the Taylor expansion.

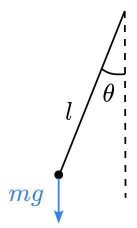
2.6 Dimensional Analysis

Dimensional analysis is a way to obtain information about solutions to equations without solving them. At a mathematical level, dimensional analysis is the ability to rescale variables to remove certain constants from equations.

Example 2.7

We will derive this equation for a pendulum later:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta. \quad \cdot 85$$



Suppose we release the pendulum from rest at some angle θ_0. We want to find the period T of oscillation where

$$\theta(t + T) = \theta(t). \quad \cdot 86$$

We can remove g, l from the equation by rescaling

$$t = \sqrt{\frac{l}{g}}\tau. \tag{87}$$

We are effectively writing $\theta(t(\tau)) = F(\tau)$ and by chain rule,

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\tau^2} = -\sin F. \tag{88}$$

This equation does not depend on g, l, and so the solution is some function $F(\tau)$. Therefore, the period $\Delta\tau$ of $F(\tau)$ may depend on the initial angle θ_0 but can't depend on g, l. Hence,

$$F(\tau + \Delta\tau) = F(\tau) \tag{89}$$

giving

$$\theta(t) = F\left(\sqrt{\frac{g}{l}}t\right) = F\left(\sqrt{\frac{g}{l}}t + \Delta\tau\right)) \tag{90}$$

$$= F\left(\sqrt{\frac{g}{l}}\left(t + \sqrt{\frac{l}{g}}\Delta\tau\right)\right) \tag{91}$$

$$= \theta\left(t + \sqrt{\frac{l}{g}}\Delta\tau\right). \tag{92}$$

Hence,

$$T = \sqrt{\frac{l}{g}}\Delta\tau. \tag{93}$$

Therefore, without solving the equation, we have found that the period of a pendulum is proportional to $\sqrt{l}$.

However, in general, a necessary rescaling may not be obvious. Thus, associating dimensions to all constants and variables is a form of bookkeeping that accounts for how these quantities appear in Newton's equations.

The basic dimensions are

- length L,
- time T,
- mass M.

Then we have

$$[\dot{x}] = LT^{-1} \tag{94}$$

$$[\ddot{x}] = LT^{-2} \tag{95}$$

$$[F] = MLT^{-2} \tag{96}$$

$$[E] = ML^2T^{-2}, \text{ etc.} \tag{97}$$

There can be other dimensions (such as charge), depending on the problem.

The fundamental principles of dimensional analysis are

- [LHS] = [RHS],
- all arguments of nontrivial functions (*i.e.* involving sums of different powers) must be dimensionless.

Example 2.8 (Pendulum, Revisited)

We first list all the dimensions of the relevant quantities:

$$[g] = LT^{-2} \tag{98}$$
$$[l] = L \tag{99}$$
$$[m] = M \tag{100}$$
$$[\theta_0] = 1 \quad \text{(dimensionless)} \tag{101}$$
$$[T] = T. \tag{102}$$

Then we let

$$T = f(\theta_0) g^A l^B m^C \tag{103}$$
$$T = L^A T^{-2A} L^B M^C \tag{104}$$

This gives

$$C = 0, A = \frac{1}{2}, B = \frac{1}{2}. \tag{105}$$

Therefore, the period is

$$T = f(\theta_0)\sqrt{\frac{l}{g}}. \tag{106}$$

> ***Remark.*** If A, B, C are not all fixed, we have a dimensionless ratio.

Lecture 5 · 2026-01-31

2.7 Friction

When objects move through a medium (*e.g.* air or water), microscopic forces between the object and the medium cause momentum to be carried off into the medium and lost.

Definition 2.9 (Friction)

Friction is a macroscopic force that keeps track of the momentum lost due to the complicated microsopic effects.

There are two important properties:

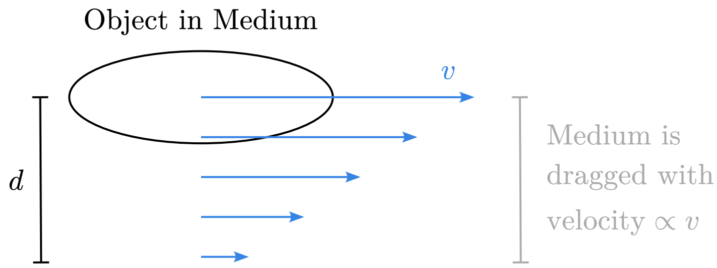
1. Friction does not conserve energy, since momentum is lost to the medium, in the form of heat.
2. Friction is irreversible. Energy is lost by the object, and energy is lost by the object but not regained. [If one every doubts about the sign of a friction force, it should slow the object

down.] Moreover, friction forces must change signs under $\mathbf{v} \to -\mathbf{v}$, hence friction forces must depend on velocity.

There are two common cases of friction forces:

1. **Linear drag.** $\boldsymbol{F} = -k_1 \boldsymbol{v}$.

   Linear drag depends on viscous effects [see IB Fluid Dynamics], such as a spoon in honey. In this case, objects move the medium with them.



Example 2.10

Stokes' law for a spherical object of radius L stats that

$$k_1 = 6\pi\eta L, \tag{107}$$

where η is the viscosity of the medium.

2. **Quadratic drag.** $\boldsymbol{F} = -k_2 \left|\boldsymbol{v}\right| \boldsymbol{v}$.

   Quadratic drag is the more intuitive case.

   As an object bumps into molecules, the rate of collisions is proportional to the speed $|\mathbf{v}|$, and each collision imparts a momentum change proportional to $|\mathbf{v}|$. Hence the force is proportional to $|\mathbf{v}|^2$.

   The number of collisions depends on the density of the medium ρ and the cross-sectional area A of the object, so $k_2 \propto \rho A$. We can also see this by dimensional analysis:

$$[F] = MLT^2 \tag{108}$$

$$\left[k_2 v^2\right] = [k_2]\left(LT^{-1}\right)^2 = [k_2]L^2T^{-2} \tag{109}$$

$$[k_2] = ML^{-1} = [\rho A]. \tag{110}$$

k_1 and k_2 are called coefficients of friction. Both linear and quadratic drag are typically present. For a spherical object, which term dominates depends on the Reynolds number R [see IB Fluid Dynamics], where

$$\frac{F_{\text{quad}}}{F_{\text{lin}}} \approx \frac{\rho A v^2}{\eta L v} = \frac{\rho v L}{\eta} \equiv R. \tag{111}$$

2.7.1 Terminal Velocity

Consider a particle falling with quadratic friction under gravity. Consider the z-component of the motion. We have

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg + kv^2. \qquad (112)$$

The velocity starts at 0, then increases. Initially, RHS is dominated by $-mg$. Eventually, the two forces balance, giving a **terminal velocity**.

$$v_{\text{term}} = -\sqrt{\frac{mg}{k}} \qquad (113)$$

so heavier objects have higher terminal velocities.

We can also consider the timescale when the object reaches its terminal velocity. With dimensional analysis, we write

$$t \propto m^A g^B k^C \qquad (114)$$

$$T = M^A \left(LT^{-2}\right)^B \left(ML^{-1}\right)^C \qquad (115)$$

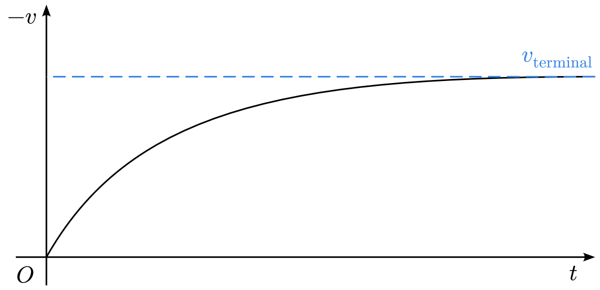
So we have

$$A = -C, \quad B = -\frac{1}{2}, \quad B = C \quad \Rightarrow \quad C = -\frac{1}{2}, A = -\frac{1}{2}. \qquad (116)$$

Therefore,

$$t \propto \sqrt{\frac{m}{kg}}. \qquad (117)$$

Note that, solving the equations gives the following graph.



We can also have motion in different directions to gravity, so that

$$m\dot{\mathbf{v}} = m\mathbf{g} - k\,|\mathbf{v}|\,\mathbf{v}. \qquad (118)$$

2.7.2 Damping

Friction damps small oscillations about an equilibrium point. For small oscillations, the linear drag dominates. Thus, in a 1D system:

$$\ddot{x} = -\omega_0^2 x - 2\alpha\dot{x} \qquad (119)$$

where ω_0 is the natural frequency of oscillations without friction, and α is the damping coefficient. Solving this gives

$$x = \mathrm{e}^{-\alpha t}\left[A_+ \mathrm{e}^{\mathrm{i}\,\Omega t} + A_- \mathrm{e}^{-\mathrm{i}\,\Omega t}\right] \qquad (120)$$

where $\Omega = \sqrt{\omega_0^2 - \alpha^2}$ and taking the real part gives the damped oscillations.

The three cases are

- $\omega_0^2 > \alpha^2$, underdamped, decaying oscillations.
- $\omega_0^2 < \alpha^2$, overdamped, exponential decay.
- $\omega_0^2 = \alpha^2$, critical damping: $x = (A + Bt)\mathrm{e}^{-\alpha t}$.

Remark. Note that $x, \ddot{x}$ are invarant under time reversal $t \to -t$, but $\dot{x} \to -\dot{x}$. As far as we know, the fundamental laws of nature are invariant under CPT. Friction forces are always off under T and cannot be fundamental forces.

Lecture 6 · 2026-02-03

3 Central Forces

An important class of potentials only depend on the distance to the origin, such that

$$V(\boldsymbol{x}) = V(|\boldsymbol{x}|) = V(r). \tag{121}$$

The force points towards (or away from) the origin, so

$$\boldsymbol{F} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\nabla r. \tag{122}$$

Recall that

$$\nabla r = \frac{\boldsymbol{x}}{r} = \hat{\boldsymbol{x}}. \tag{123}$$

Thus,

$$\boldsymbol{F} = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\boldsymbol{x}} \tag{124}$$

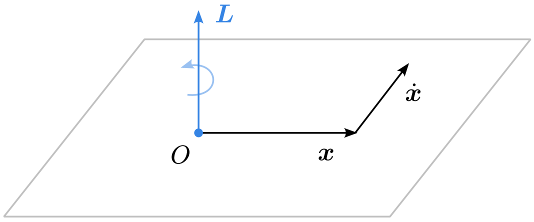
We will study the motion of a particle under a central force, particularly in polar coordinates.

3.1 Conservation of Angular Momentum

The most important fact about central potentials is that angular momentum is conserved. We have

$$\boldsymbol{L} = m\boldsymbol{x} \times \dot{\boldsymbol{x}} = \boldsymbol{x} \times \boldsymbol{p}, \tag{125}$$

where $\boldsymbol{L}$ is the angular momentum, and $\boldsymbol{p} = m\dot{\boldsymbol{x}}$ is the linear momentum. Note that $\boldsymbol{L}$ is orthogonal to both position and velocity/momentum.



$\boldsymbol{L}$ is defined relative to an origin, here we are setting the origin at $\boldsymbol{x} = \boldsymbol{0}$, which will be generalised later.

For a general force $\boldsymbol{F}$,

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = m\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x} \times \dot{\boldsymbol{x}}) = m(\dot{\boldsymbol{x}} \times \dot{\boldsymbol{x}} + \boldsymbol{x} \times \ddot{\boldsymbol{x}}) = \boldsymbol{x} \times \boldsymbol{F} \equiv \boldsymbol{G}, \tag{126}$$

where $\boldsymbol{G}$ is the torque. *i.e.*

$$\dot{\boldsymbol{L}} = \boldsymbol{G}. \tag{127}$$

This is analogous to Newton's law Eq · 8, but for rotational motion. [$\boldsymbol{G}$ can be thought of as the *rotational force*, and $\boldsymbol{L}$ as the *rotational momentum*.]

For a central force, $\boldsymbol{F} \parallel \hat{\boldsymbol{x}} \Rightarrow \boldsymbol{x} \times \boldsymbol{F} = \boldsymbol{G} = \boldsymbol{0}$. Thus, angular momentum is conserved:

$$\dot{\boldsymbol{L}} = \mathbf{0}. \tag{128}$$

Since $\boldsymbol{L}$ doesn't change, and obeys

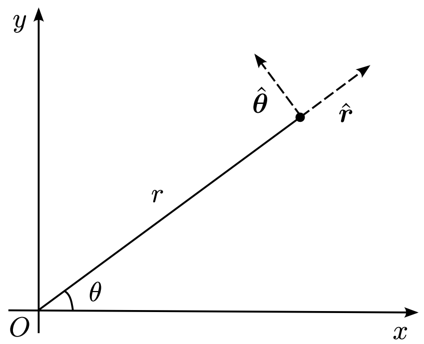
$$\begin{cases} \boldsymbol{L} \cdot \boldsymbol{x} = 0 \\ \boldsymbol{L} \cdot \dot{\boldsymbol{x}} = 0 \end{cases} \tag{129}$$

where the position and velocity are constrained to a plane perpendicular to $\boldsymbol{L}$. Hence, we have reduced the problem from 3D to 2D.

3.2 Polar Coordinates in the Plane

Within polar coordinates, we have

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{130}$$



In cartesian coordinates, [see IA Vector Calculus for more details]

$$\hat{\boldsymbol{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}. \tag{131}$$

Note that

$$\hat{\boldsymbol{r}}^2 = \hat{\boldsymbol{\theta}}^2 = 1, \quad \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\theta}} = 0. \tag{132}$$

> ***Important.*** These vectors depend on positions. We have
>
> $$\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}\theta} = \hat{\boldsymbol{\theta}}, \quad \frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}\theta} = -\hat{\boldsymbol{r}}. \tag{133}$$
>
> Hence, we must keep track of these changes when we write equations in polar coordinates.

Consider Newton's equation [*i.e.* $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$] in polar coordinates,

$$\boldsymbol{x} = r\hat{\boldsymbol{r}} \tag{134}$$

$$\dot{\boldsymbol{x}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}} \tag{135}$$

$$= \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}\theta} \tag{136}$$

$$= \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \tag{137}$$

$$\ddot{\boldsymbol{x}} = \ddot{r}\hat{\boldsymbol{r}} + 2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\boldsymbol{r}} \tag{138}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}} \tag{139}$$

Example 3.1 (Circular Motion at Constant Angular Speed)

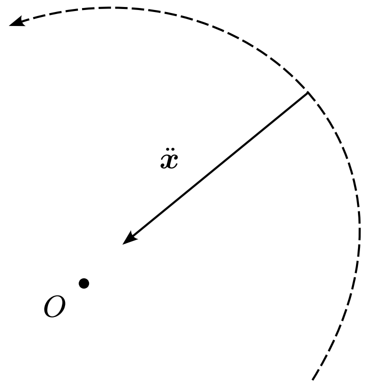
We have

$$\dot{r} = 0, \quad \ddot{r} = 0, \quad \dot{\theta} = \omega, \quad \ddot{\theta} = 0. \tag{140}$$

Thus,

$$\ddot{\boldsymbol{x}} = -r\omega^2 \hat{\boldsymbol{r}}. \tag{141}$$

Note that circular motion requires a centripetal force towards the origin.



Matching components of Eq · 124 with Eq · 139 substituted in Eq · 8, we have

$$\hat{\boldsymbol{\theta}}: \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0, \tag{142}$$

$$\hat{\boldsymbol{r}}: \quad m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\mathrm{d}V}{\mathrm{d}r}. \tag{143}$$

These are the Newton's equations for a central force in polar coordinates.

Hence, Eq · 142 gives

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0. \tag{144}$$

Thus $l \equiv r^2\dot{\theta}$ is constant.

In fact, this is the magnitude of the angular momentum per unit mass:

$$\boldsymbol{L} = m\boldsymbol{x} \times \dot{\boldsymbol{x}} \tag{145}$$

$$= mr\hat{\boldsymbol{r}} \times \left(\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right) \tag{146}$$

$$= mr^2\dot{\theta}\left(\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}}\right) \tag{147}$$

$$|\boldsymbol{L}| = mr^2\dot{\theta} = ml. \tag{148}$$

Sometimes, l is called the "angular momentum", even though it is angular momentum per unit mass. Using the definition of l in the equation for $\hat{r}$, we have

$$m\left(\ddot{r} - r\frac{l^2}{r^4}\right) = -\frac{\mathrm{d}V}{\mathrm{d}r}. \tag{149}$$

We can rewrite this as

$$m\ddot{r} = -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} \tag{150}$$

where V_{eff} is the effective potential defined by

$$V_{\text{eff}}(r) = V(r) + \frac{ml^2}{2r^2}. \tag{151}$$

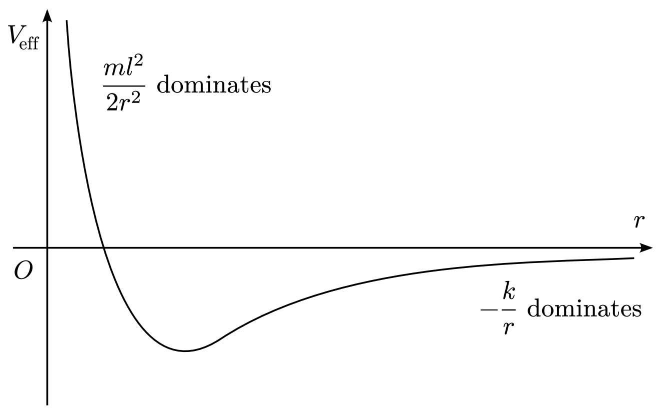
Hence we have reduced the motion to an effective one-dimensional problem in the radial direction. This is possible because of the conservation of angular momentum.

> ***Remark.*** The effective potential is what a radial observer would see. There seems to be an extra repulsive potential at small r due to the angular momentum, which is called the centrifugal barrier.

3.3 The Effective Potential

Consider $V(r) = -\frac{k}{r}$, *i.e.* an attractive $\frac{1}{r}$ potential. The effective potential is

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{ml^2}{2r^2}. \tag{152}$$



We have a centrifugal barrier at small r due to the angular momentum term. The angular momentum prevents the particle from getting too close to the origin.

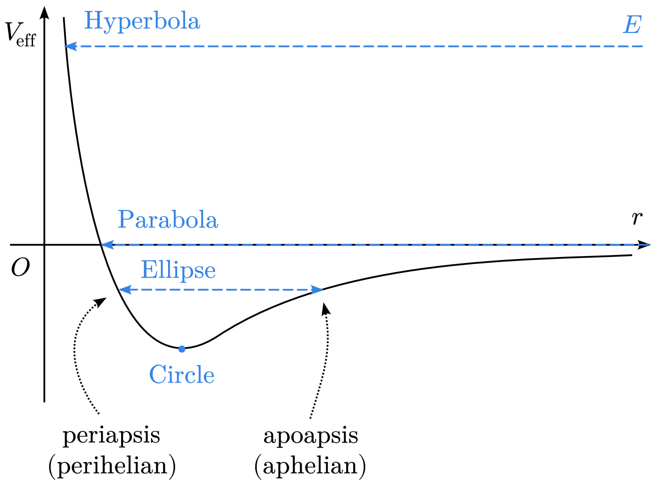
We can also see the effective potential from the conserved energy,

$$\begin{aligned} E &= \frac{1}{2}m\dot{\mathbf{x}}^2 + V(r) && \cdot\ 153 \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) && \cdot\ 154 \\ &= \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2} + V(r) && \cdot\ 155 \\ &= \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r). && \cdot\ 156 \end{aligned}$$

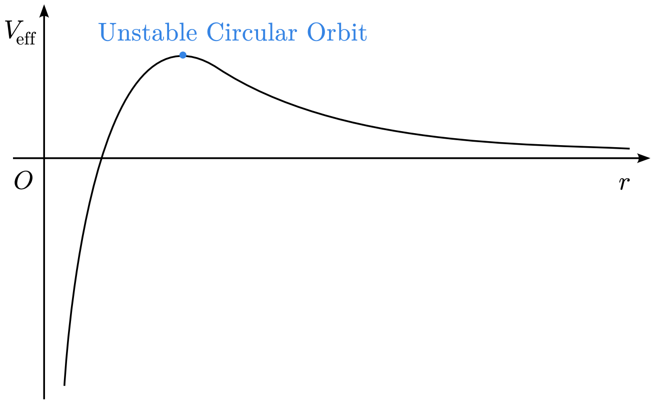
The centrifugal barrier is the angular kinetic energy.

Different cases arise depending on the energy E.

Lecture 7 · 2026-02-05



Now, suppose instead that $V = -\frac{k}{r^n}$ where $n > 2$. Then



Note that in $V = -\frac{k}{r^n}$ there are no stable bound orbits, and the particle can fall to the origin.

Remark. Gravity in d space dimensions has

$$V \propto \frac{1}{r^{d-2}}. \tag{157}$$

Moreover, circular orbits are stable only for $d < 4$. Hence, our universe has a special number of dimensions for stable planetary orbits.

3.4 The Orbit Equation

We shall now see how to solve the equations of motion for a central potential. Consider

$$u = \frac{1}{r}. \tag{158}$$

We want to derive the orbit equation for $u(\theta)$. Recall the case that

$$m\ddot{r} = -\frac{dV_{\text{eff}}}{dr} \tag{159}$$

$$V_{\text{eff}}(r) = V(r) + \frac{ml^2}{2r^2} \tag{160}$$

$$V(r) = -\frac{k}{r}. \tag{161}$$

Under the change of variables $r(t) \to u(\theta)$,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{l}{r^2} = -l\frac{\mathrm{d}u}{\mathrm{d}\theta} \quad \cdot 162$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-l\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = -l\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -l^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}. \quad \cdot 163$$

Then

$$m\ddot{r} - \frac{ml^2}{r^3} = F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r} \quad \cdot 164$$

$$-l^2mu^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - ml^2u^3 = F\left(\frac{1}{u}\right) \quad \cdot 165$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{1}{ml^2u^2}F\left(\frac{1}{u}\right). \quad \cdot 166$$

A special case arises when $V = -\frac{km}{r}$, *i.e.* the Kepler problem.

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{k}{l^2}. \quad \cdot 167$$

This equation is a harmonic oscillator with a displaced centre. The solution is

$$u = A\cos(\theta - \theta_0) + \frac{k}{l^2}. \quad \cdot 168$$

Note that u is largest at $\theta = \theta_0$, and r is smallest there, which is the periapsis.

We can choose axes on the plane so that $\theta_0 = 0$. Then

$$r = \frac{r_0}{e\cos\theta + 1} \quad \cdot 169$$

where $r_0 = \frac{l^2}{k}$ and e is a constant of integration. The shape of the orbit depends on e. Note that this is the equation for a conic section in polar coordinates, and e is the eccentricity.

$$\left(\text{bounded orbit}, \frac{r}{r_0} \in \left[\frac{1}{1+e}, \frac{1}{1-e}\right]\right) \quad E < 0 \qquad e = 0 : \text{circular orbit} \quad \cdot 170$$

$$0 < e < 1 : \text{elliptical orbit} \quad \cdot 171$$

$$E = 0 \qquad e = 1 : \text{parabolic orbit} \quad \cdot 172$$

$$E > 0 \qquad e > 1 : \text{hyperbolic orbit} \quad \cdot 173$$

Proof. [Ellipse.]

With $0 < e < 1$, we can rearrange to get

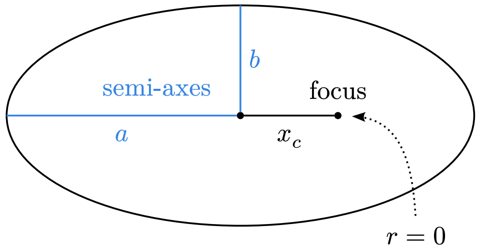
$$r_0 - re\cos\theta = r \quad \cdot 174$$

$$(r_0 - re\cos\theta)^2 = r^2 \quad \cdot 175$$

$$(r_0 - ex)^2 = x^2 + y^2. \quad \cdot 176$$

Hence we can regroup to get

$$\frac{(x - x_c)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{177}$$



where a, b, x_c are given in terms of e and r_0. For example,

$$x_c = -\frac{e r_0}{1 - e^2} = -ea. \tag{178}$$

Planets in the solar system have small e, so that they are close to circular. *e.g.*

- the largest e is for Mercury, which has $e \approx 0.2$.
- Halley's comet has $e \approx 0.97$.

Proof. [Parabola.]

With $e = 1$, we get

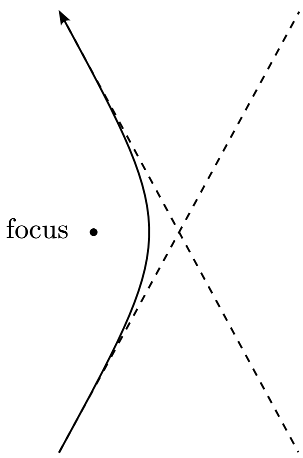
$$r_0^2 - 2r_0 x + x^2 = x^2 + y^2 \tag{179}$$

$$r_0^2 - 2r_0 x = y^2. \tag{180}$$

This is the equation for a parabola.

Proof. [Hyperbola.]

Note that $r \to \infty$ at $\cos\theta = -\frac{1}{e}$. Hence, the asymptotes satisfy $\theta > \frac{\pi}{2}$. Hence,



We can evaluate the energy on the solution,

$$E = \frac{1}{2} m \dot{r}^2 + \frac{m l^2}{2r^2} - \frac{km}{r} \tag{181}$$

$$= \frac{mk^2}{2l^2}\left(e^2 - 1\right) \quad \text{by algebra.} \tag{182}$$

Hence the energy is negative for bounded orbits, zero for parabolic orbits, and positive for hyperbolic orbits. In particular, for a circular orbit,

$$E = -\frac{mk^2}{2l^2} \tag{183}$$

which is the minimum of $V(r)$.

3.5 Kepler's Laws

A consequence of the above are Kepler's laws of planetary motion.

Proposition 3.2 (Kepler's Laws)

K1 Planets move in ellipses with the Sun at one focus.

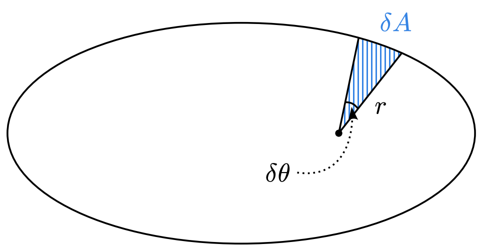
K2 The line between the planet and the Sun sweeps out equal areas in equal times, and

$$\dot{A} = \frac{l}{2}. \tag{184}$$

K3 The period of of the orbit is proportional to radius$^{\frac{3}{2}}$.

Proof.

K2 We have



where

$$\delta A = \frac{1}{2}r^2\delta\theta \Rightarrow \dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2}. \tag{185}$$

It follows from the conservation of angular momentum, for any central force.

Lecture 8 · 2026-02-10

K3 It is natural to consider dimensional analysis. The only parameter in Newton's equation is $k = GM$. Hence the period satisfies

$$T = cR^A k^B \tag{186}$$

$$[T] = [R]^A [k]^B \tag{187}$$

$$T = L^A\left(L^3T^{-2}\right)^B = L^{A+3B}T^{-2B}. \tag{188}$$

Hence $A + 3B = 0$ and $-2B = 1$, so $A = \frac{3}{2}$ and $B = -\frac{1}{2}$. Thus,

$$T = c\frac{R^{\frac{3}{2}}}{k^{\frac{1}{2}}}. \quad (189)$$

Note that there is no unique radius associated to an ellipse, but taking any will do.

More precisely, starting with $\dot{A} = \frac{l}{2}$, we have the full period

$$T = \int_0^T \mathrm{d}t \quad (190)$$

$$= \int_0^A \frac{2}{l}\,\mathrm{d}A \quad (191)$$

$$= \frac{2}{l}A \quad (192)$$

$$= \frac{2}{l}\pi ab \quad (193)$$

$$= \frac{2\pi}{l}\frac{r_0^2}{(1-e^2)^{\frac{3}{2}}} \quad (194)$$

$$= \frac{2\pi}{\sqrt{GM}}\left(\frac{r_0}{1-e^2}\right)^{\frac{3}{2}} \quad \text{by } l^2 = kr_0 \quad (195)$$

$$= \frac{2\pi}{\sqrt{GM}}R_{\text{avg}}^{3/2} \quad (196)$$

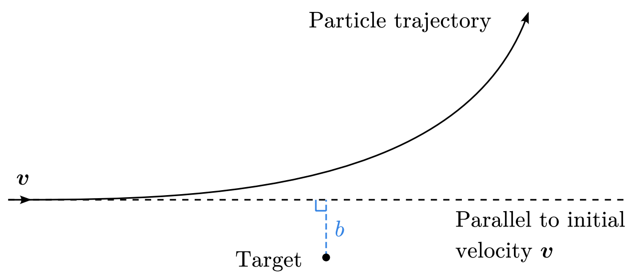
where $R_{\text{avg}} = \frac{r_0}{1-e^2}$ is the average radius of the ellipse.

3.6 Repulsive Potentials and Scattering

Given a central potential $V(r)$ such that $V \to 0$ as $r \to \infty$, one can perform scattering experiments by sending in a particle from large r and see how it moves out again.

Definition 3.3 (Impact Parameter)

The **impact parameter** b is the distance of closest approach if there were no forces.



Proposition 3.4

The impact parameter is related to the angular momentum (per unit mass) $l = \frac{|L|}{m}$ as

$$l = bv. \quad (197)$$

Proof. A non-interacting particle has a conserved angular momentum. The velocity does not change. At the closest point,

$$l = |\mathbf{x} \times \dot{\mathbf{x}}| = bv. \quad \cdot 198$$

This must also be the angular momentum at the start. But the initial l is the same for the interacting and non-interacting particles and is also conserved in the interacting case.

Rutherford scattering (1911) showed that certain scattering experiments of atoms could be explained if all the positive charge in an atom was confined to a tiny nucleus.

Scattering by a repulsive interaction for two positively charged parciules satisfies

$$V = \frac{\kappa}{r} \quad \text{with } \kappa = \frac{qQ}{4\pi\varepsilon_0}. \quad \cdot 199$$

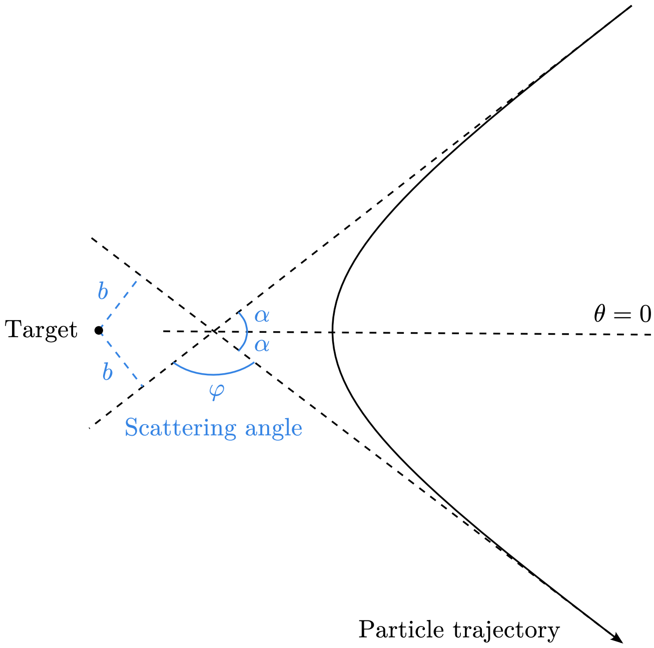
We may reuse results from Kepler problem by setting $-km \to \kappa$.

In particular, the orbits are

$$r = \frac{r_0}{\tilde{e}\cos\theta - 1} \quad \cdot 200$$

with $r_0 = \frac{l^2 m}{\kappa}$.

We want to find the angle φ through which the particle is scattered.



Note that incoming and outgoing impact parameters are equal by conservation of l and E. [And hence v at infinity.]

Clearly

- $\pi = \varphi + 2\alpha$
- $\tilde{e}\cos\alpha = 1$ [The particle goes to infinity at $\theta = \alpha$.]

Then we can get φ in terms of impact parameter and initial velocity,

$$E = \frac{1}{2}mv^2 \qquad \text{conserved, using intial energy} \tag{201}$$

$$= \frac{\kappa^2}{2l^2m}\left(\tilde{e}^2 - 1\right) \qquad \text{same formula as for conic sections} \tag{202}$$

$$= \frac{\kappa^2}{2mb^2v^2}\tan^2\alpha \qquad l = bv \text{ and } \left(\tilde{e}^2 - 1\right) = \frac{1}{\cos^2\alpha} - 1 \tag{203}$$

$$= \frac{\kappa^2}{2mb^2v^2}\frac{1}{\tan^2\frac{\varphi}{2}} \qquad \tan\alpha = \tan\frac{\pi - \varphi}{2} = \frac{1}{\tan\frac{\varphi}{2}} \tag{204}$$

Hence, matching the first and last expressions, we have

$$\varphi = 2\arctan\frac{\kappa}{mbv^2}. \tag{205}$$

Note that a small b leads to large angle scattering, and it allows scattering to probe very small distances.

4 Systems of Particles

There are many particles in the universe. We shall focus on N of them.

Label the particles by $i = 1, ..., N$. Each particle has a momentum

$$\boldsymbol{p}_i = m_i \dot{\boldsymbol{x}}_i \quad \cdot 206$$

and obeys Newton's law individually,

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i. \quad \cdot 207$$

The force $\boldsymbol{F}_i$ on the i-th particle can be external or due to the other particles:

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{\text{ext}} + \sum_{j \neq i} \boldsymbol{F}_{ij}, \quad \cdot 208$$

where $\boldsymbol{F}_{ij}$ is the force on the i-th particle due to the j-th particle. The forces between particles are found to obey

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji} \quad \cdot 209$$

due to Newton's third law.

Law 4.1 (Newton's Third Law)

Every action has an equal and opposite reaction.

Lecture 9 · 2026-02-12

4.1 Centre of Mass

Consider a system of N particles.

Definition 4.2 (Centre of Mass)

The total mass of a system is $M = \sum_{i=1}^{N} m_i$.

The centre of mass of a system is defined as

$$\boldsymbol{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \boldsymbol{x}_i. \quad \cdot 210$$

The total momentum of a system can therefore be expressed as

$$\boldsymbol{P} = \sum_{i=1}^{N} \boldsymbol{p}_i \quad \cdot 211$$

$$= \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \quad \cdot 212$$

$$= M \dot{\boldsymbol{R}}. \quad \cdot 213$$

Therefore, the centre of mass moves as if it is a single particle of mass M.

4.1.1 Conservation of Momentum

We have

$$\dot{P} = \sum_i \dot{p}_i \tag{214}$$

$$= \sum_i \left(F_i^{\text{ext}} + \sum_{j \neq i} F_{ij} \right) \tag{215}$$

$$= \sum_i F_i^{\text{ext}} + \sum_{i<j} \underbrace{\left(F_{ij} + F_{ji} \right)}_{=0 \text{ by N3}} \tag{216}$$

$$= \sum_i F_i^{\text{ext}}. \tag{217}$$

We have hence shown that

$$M\ddot{R} = F = \sum_i F_i^{\text{ext}}. \tag{218}$$

Hence, the centre of mass accelerates like a point particle, subject to an external force F.

> ***Remark.*** This is the reason why we can treat the Earth as a point particle, since its internal forces cancel out.

In particular, $F = 0$ implies that $\dot{P} = 0$, so the total momentum of a system is conserved in the absence of external forces.

4.1.2 Conservation of Angular Momentum

A similar result holds for total angular momentum. Consider, about a fixed point a,

$$L = \sum_i (x_i - a) \times p_i \tag{219}$$

$$\dot{L} = \sum_i (x_i - a) \times \dot{p}_i \tag{220}$$

$$= \sum_i (x_i - a) \times \left(F_i^{\text{ext}} + \sum_{j \neq i} F_{ij} \right) \tag{221}$$

$$= \underbrace{\sum_i (x_i - a) \times F_i^{\text{ext}}}_{G} + \sum_{i<j} \left((x_i - a) \times F_{ij} + \left(x_j - a\right) \times F_{ji} \right) \tag{222}$$

$$= G + \sum_{i<j} \left(x_i - x_j\right) \times F_{ij} \tag{223}$$

where G is the total external torque on the system about a. Note that the final term does not vanish in general. However, if the force comes from a potential that depends on the distance from x_i to x_j then

$$F_{ij} = -\nabla_{x_i} V\left(\left|x_i - x_j\right|\right) = -V'\left(\left|x_i - x_j\right|\right) \frac{x_i - x_j}{\left|x_i - x_j\right|}, \tag{224}$$

so

$$\left(\boldsymbol{x}_i - \boldsymbol{x}_j\right) \times \boldsymbol{F}_{ij} = 0. \quad \cdot 225$$

In this case, we have $\dot{\boldsymbol{L}} = \boldsymbol{G}$, so the total angular momentum is conserved in the absence of external torques.

It is more subtle to prove for other forces such as the Lorentz force, but it can be shown that the total angular momentum is still conserved for all known forces. So,

$$\dot{\boldsymbol{L}} = \boldsymbol{G}. \quad \cdot 226$$

We often take $\boldsymbol{a} = \boldsymbol{R}$, the centre of mass. However in general, in a general $\boldsymbol{R}(t)$, we need to generalise the above definition (refer to Example Sheet 3).

4.1.3 Conservation of Energy

We can write

$$\boldsymbol{x}_i = \boldsymbol{R} + \boldsymbol{y}_i \quad \cdot 227$$

where $\boldsymbol{y}_i$ is the position of the i-th particle relative to the centre of mass. We have

$$M\boldsymbol{R} = \sum_i m_i \boldsymbol{x}_i \Rightarrow \sum_i m_i \boldsymbol{y}_i = \boldsymbol{0}. \quad \cdot 228$$

Note that this is true for all time, so

$$\sum_i m_i \dot{\boldsymbol{y}}_i = \boldsymbol{0}. \quad \cdot 229$$

The kinetic energy is

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i \dot{\boldsymbol{x}}_i \cdot \dot{\boldsymbol{x}}_i && \cdot 230 \\ &= \sum_i \frac{1}{2} m_i \left(\dot{\boldsymbol{R}}^2 + \dot{\boldsymbol{y}}_i^{\,2} + 2\dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{y}}_i\right) && \cdot 231 \\ &= \underbrace{\frac{1}{2} M \dot{\boldsymbol{R}}^2}_{\text{CoM Kinetic Energy}} + \underbrace{\sum_i \frac{1}{2} m_i \dot{\boldsymbol{y}}_i^{\,2}}_{\text{internal kinetic energy}}. && \cdot 232 \end{aligned}$$

To have a conserved total energy, all forces must be conservative:

$$\begin{aligned} \boldsymbol{F}_i^{\text{ext}} &= -\boldsymbol{\nabla}_i V_i(\boldsymbol{x}_i) && \cdot 233 \\ \boldsymbol{F}_{ij} &= -\boldsymbol{\nabla}_i V_{ij}\left(\left|\boldsymbol{x}_i - \boldsymbol{x}_j\right|\right). && \cdot 234 \end{aligned}$$

To obey N3, we must have $V_{ij} = V_{ji}$. Hence one can show that

$$E = T + \sum_i V_i(\boldsymbol{x}_i) + \sum_{i<j} V_{ij}\left(\left|\boldsymbol{x}_i - \boldsymbol{x}_j\right|\right) \quad \cdot 235$$

is a conserved quantity, *i.e.* $\dot{E} = 0$. To check this,

$$\dot{E} = \sum_i m_i \dot{\boldsymbol{x}}_i \cdot \ddot{\boldsymbol{x}}_i + \sum_i \boldsymbol{\nabla}_i V_i \cdot \dot{\boldsymbol{x}}_i + \sum_{i<j} \left[\boldsymbol{\nabla}_i V_{ij} \cdot \dot{\boldsymbol{x}}_i + \boldsymbol{\nabla}_j V_{ij} \cdot \dot{\boldsymbol{x}}_j\right] \quad \cdot 236$$

$$= \sum_i m_i \dot{\boldsymbol{x}}_i \cdot \ddot{\boldsymbol{x}}_i - \sum_i \boldsymbol{F}_i^{\text{ext}} \cdot \dot{\boldsymbol{x}}_i - \sum_{i<j} \left[\boldsymbol{F}_{ij} \cdot \dot{\boldsymbol{x}}_i + \boldsymbol{F}_{ji} \cdot \dot{\boldsymbol{x}}_j\right] \quad \cdot 237$$

$$= \sum_i \dot{\boldsymbol{x}}_i \cdot \left(m_i \ddot{\boldsymbol{x}}_i - \boldsymbol{F}_i^{\text{ext}} - \sum_{j \neq i} \boldsymbol{F}_{ij}\right) \quad \cdot 238$$

$$= 0. \quad \cdot 239$$

Lecture 10 · 2026-02-14

4.2 The Two Body Problem

An important special case is the two body problem, where there are two particles with no external forces (*e.g.* the Earth and the Moon). We can reduce this to a single particle problem by working in terms of relative separation. Let $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ be the relative separation.

The centre of mass is given by

$$M\boldsymbol{R} = m_1 \boldsymbol{x}_1 + m_2 \boldsymbol{x}_2. \quad \cdot 240$$

So we have

$$\boldsymbol{x}_1 = \boldsymbol{R} + \frac{m_2}{M}\boldsymbol{r} \quad \cdot 241$$

$$\boldsymbol{x}_2 = \boldsymbol{R} - \frac{m_1}{M}\boldsymbol{r} \quad \cdot 242$$

By Eq · 232, the kinetic energy is

$$T = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}m_1\frac{m_2^2}{M^2}\dot{\boldsymbol{r}}^2 + \frac{1}{2}m_2\frac{m_1^2}{M^2}\dot{\boldsymbol{r}}^2 \quad \cdot 243$$

$$= \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 \quad \cdot 244$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \cdot 245$$

is called the **reduced mass**.

Note that

$$\mu\ddot{\boldsymbol{r}} = \mu(\ddot{\boldsymbol{x}}_1 - \ddot{\boldsymbol{x}}_2) \quad \cdot 246$$

$$= \mu\left(\frac{\boldsymbol{F}_{12}}{m_1} - \frac{\boldsymbol{F}_{21}}{m_2}\right) \quad \cdot 247$$

$$= \mu\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\boldsymbol{F}_{12} \quad \text{since } \boldsymbol{F}_{12} = \boldsymbol{F}_{21} \quad \cdot 248$$

$$= \boldsymbol{F}_{12} \quad \cdot 249$$

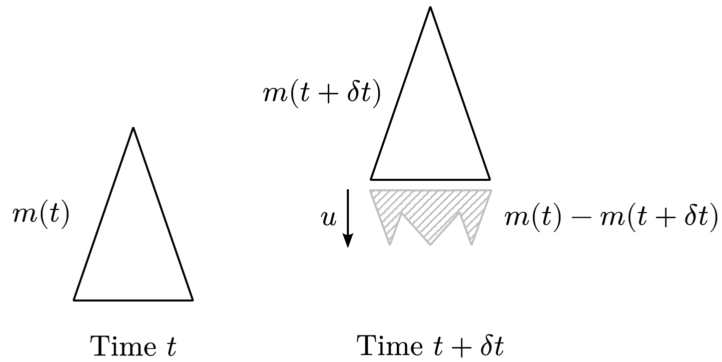
Therefore, relative separation also behaves like a single particle problem, and we can use methods already developed.

If one mass is much larger than the other, say $m_1 \gg m_2$, then $\mu \approx \frac{m_1 m_2}{m_1 + m_2} \approx m_2$. *i.e.* in this limit, heavy object is essentially still and the lighter object moves around it.

In general, for $N > 2$, we cannot solve the problem analytically. However, if $N \gg 1$, we can use statistical physics to make progress (see Part II).

4.3 Rocket Equation and Variable Mass

The relation $\dot{\boldsymbol{P}} = \boldsymbol{F}^{\text{ext}}$ is useful when the internal forces are complicated. For example, consider a rocket that ejects fuel with speed u relative to the rocket.



We shall work in the z-axis, since the rocket moves in a straight line.

The process of ejection can be complicated, but the total momentum must obey

$$\dot{P} = \frac{P(t+\delta t) - P(t)}{\delta t} = F^{\text{ext}}. \tag{250}$$

Let $v(t)$ be the speed of the rocket at time t. We have

$$P(t) = m(t)v(t) \tag{251}$$

$$P(t+\delta t) = \underbrace{m(t+\delta t)v(t+\delta t)}_{\text{rocket}} + \underbrace{(m(t) - m(t+\delta t))(v(t) - u)}_{\text{fuel}} \tag{252}$$

$$= m(t)v(t) + [m'(t)v(t) + m(t)v'(t)]\delta t - m'(t)(v(t) - u)\delta t \tag{253}$$

Hence

$$P(t+\delta t) - P(t) = (m\dot{v} + u\dot{m})\delta t \tag{254}$$

$$\Rightarrow m\dot{v} + u\dot{m} = F^{\text{ext}}. \tag{255}$$

This is called the Tsiolkovsky rocket equation. We can also write this as

$$m\dot{v} = F^{\text{ext}} - u\dot{m}. \tag{256}$$

where $-u\dot{m}$ is the thrust force, coming from N3.

5 Rigid Bodies

This is a class of tractable N-body problems, where the distances between the N particles are fixed. In practice, this is due to very strong internal forces.

Definition 5.1 (Rigid Body)

A **rigid body** is a collection of particles such that the distance between any two particles is fixed.

The only motions a rigid body can undergo are translations of the centre of mass, and rotations.

5.1 Angular Velocity

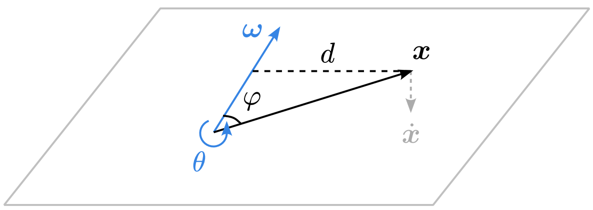
In three dimensions, rotations are described by an angular velocity vector $\boldsymbol{\omega}$. We write

$$\boldsymbol{\omega} = \omega\hat{\boldsymbol{n}}, \quad \cdot 257$$

where $\hat{\boldsymbol{n}}$ points along the axis of rotation, and $\omega = |\boldsymbol{\omega}| = \dot{\theta}$ is the angular speed of rotation. The direction of $\boldsymbol{\omega}$ is determined by the right hand rule.

This is captured by the equation

$$\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x}. \quad \cdot 258$$



We have

- $\dot{\boldsymbol{x}}$ orthogonal to both $\boldsymbol{\omega}$ and $\boldsymbol{x}$
- $|\dot{\boldsymbol{x}}| = \omega|\boldsymbol{x}|\sin\varphi = \omega d$

Hence indeed $\omega = \left|\dot{\theta}\right|$. Note that $d = |\hat{\boldsymbol{n}} \times \boldsymbol{x}|$.

In addition to the angular velocity, a rotation must specify a point about which the axis of ration passes, since there are infinitely many parallel axis an object can rotate about.

$\boldsymbol{x}$ in the equation $\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x}$ is the position relative to some (any) point on the axis of rotation.

Remark. The equations above instantaneous behavior of a particle, so $\boldsymbol{\omega}$ can depend on t.

Lecture 11 · 2026-02-17

5.2 Moment of Inertia

Rotation of a particle involves kinetic energy. For a single particle, we have

$$T = \frac{1}{2}m\dot{\boldsymbol{x}}^2 = \frac{1}{2}m(\boldsymbol{\omega} \times \boldsymbol{x})^2 \quad \cdot 259$$

$$= \frac{1}{2} m\omega^2 d^2 \qquad \cdot 260$$

where $d = |\hat{\boldsymbol{n}} \times \boldsymbol{x}|$ is the perpendicular distance of particle from an axis of rotation.

In a rigid body, all particles rotate with the same angular velocity:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{\omega} \times \boldsymbol{x}_i. \qquad \cdot 261$$

This keeps the distances between particles fixed, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\boldsymbol{x}_i - \boldsymbol{x}_j\right|^2 = 2\left(\dot{\boldsymbol{x}}_i - \dot{\boldsymbol{x}}_j\right) \cdot \left(\boldsymbol{x}_i - \boldsymbol{x}_j\right) = 2\left(\boldsymbol{\omega} \times \left(\boldsymbol{x}_i - \boldsymbol{x}_j\right)\right) \cdot \left(\boldsymbol{x}_i - \boldsymbol{x}_j\right) = 0. \qquad \cdot 262$$

The kinetic energy of a rigid body is then

$$T = \frac{1}{2}\sum_i m_i \dot{\boldsymbol{x}}_i^{\,2} = \frac{1}{2} I\omega^2, \qquad \cdot 263$$

where

$$I = \sum_i m_i d_i^2 \qquad \cdot 264$$

is the **moment of inertia** of the rigid body about the axis of rotation.

> ***Remark.*** I depends on the choice of axis of rotation, since d_i does.

In Eq · 263, see that I is effectively a *rotational mass*. The bigger I is, the harder it is to rotate the body.

The **angular momentum** of a rigid body is

$$\boldsymbol{L} = \sum_i m_i \boldsymbol{x}_i \times \dot{\boldsymbol{x}}_i \qquad \cdot 265$$

$$= \sum_i m_i \boldsymbol{x}_i \times (\boldsymbol{\omega} \times \boldsymbol{x}_i). \qquad \cdot 266$$

In this course, we only consider the component of $\boldsymbol{L}$ along the axis of rotation, so define

$$L = \boldsymbol{L} \cdot \hat{\boldsymbol{n}} \qquad \cdot 267$$

$$= \omega \sum_i m_i [\boldsymbol{x}_i \times (\hat{\boldsymbol{n}} \times \boldsymbol{x}_i)] \cdot \hat{\boldsymbol{n}} \qquad \cdot 268$$

$$= \omega \sum_i m_i (\hat{\boldsymbol{n}} \times \boldsymbol{x}_i) \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{x}_i) \qquad \cdot 269$$

$$= \omega \sum_i m_i d_i^2 \qquad \cdot 270$$

$$= \omega I. \qquad \cdot 271$$

Again, we can observe that I is a *rotational mass*.

Recall that torque causes change in the angular momentum, as $\dot{\boldsymbol{L}} = \boldsymbol{G}$. If the torque is also along the axis of rotation, then we can write

$$\boldsymbol{G} = G\hat{\boldsymbol{n}}, \qquad \cdot 272$$

and dotting $\dot{\boldsymbol{L}} = \boldsymbol{G}$ with $\hat{\boldsymbol{n}}$ gives

$$G = I\dot{\omega}. \tag{273}$$

Hence G acts like a *rotational force*, causing change in the angular velocity.

To calculate the moment of inertia, we use the fact that at large N, the particles are densely spaced, and the sums can be approximated by integrals.

$$\sum_i m_i f(\mathbf{x}_i) \approx \int f(\mathbf{x})\rho(\mathbf{x})\mathrm{d}^3\mathbf{x} \tag{274}$$

where $\rho(\mathbf{x})$ is the density of mass of the rigid body. We typically consider uniform density, so

$$\rho(x) = \rho_0, \tag{275}$$

which is a constant.

For example, we have

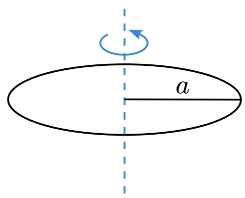
$$M = \sum_i m_i = \int \rho(\mathbf{x})\mathrm{d}^3\mathbf{x} \tag{276}$$

$$I = \int \rho(\mathbf{x})x_\perp^2\mathrm{d}^3\mathbf{x} \tag{277}$$

where $x_\perp$ is the perpendicular distance from $\mathbf{x}$ to the axis of rotation.

Example 5.2 (Moment of Inertia of Rigid Bodies With Uniform Density)

1. Consider a rotating hoop of radius a. We have

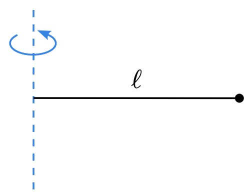


$$M = 2\pi a\rho \tag{278}$$

$$I = 2\pi a^3\rho \tag{279}$$

Hence $I = Ma^2$.

2. Consider a rotating rod of length ℓ about an axis through an endpoint.



Then

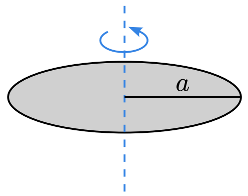
$$M = \ell\rho \tag{280}$$

$$I = \rho\int_0^\ell x^2\,\mathrm{d}x \tag{281}$$

$$= \frac{1}{3}\ell^3\rho \tag{282}$$

Hence $I = \frac{1}{3}M\ell^2$.

3. Consider a rotating disc of radius a about an axis through its centre.



Then

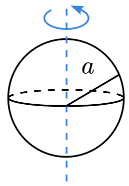
$$M = \pi a^2 \rho \tag{283}$$

$$I = \rho \int_0^a 2\pi r^3 \, \mathrm{d}r \tag{284}$$

$$= \frac{1}{2}\pi a^4 \rho \tag{285}$$

Hence $I = \frac{1}{2}Ma^2$.

4. Consider a rotating sphere of radius a about an axis through its centre.



Then

$$M = \frac{4}{3}\pi a^3 \rho \tag{286}$$

$$I = \rho \int_0^a 4\pi r^4 \, \mathrm{d}r \int_0^\pi \sin^3\theta \, \mathrm{d}\theta \tag{287}$$

$$= \frac{8}{15}\pi a^5 \rho \tag{288}$$

Hence $I = \frac{2}{5}Ma^2$.

5.3 Perpendicular Axis Theorem

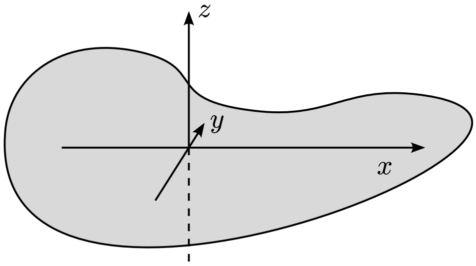
We will now consider less symmetric axes.

Theorem 5.3 (Perpendicular Axis Theorem)

For a planar body, the moment of inertia about an axis perpendicular to the plane is the sum of the moments of inertia about any two orthogonal axes in the plane.

Proof.

Consider any 2D body.



Then

$$I_z = \int \rho\left(x^2 + y^2\right)\mathrm{d}^2x \quad (289)$$

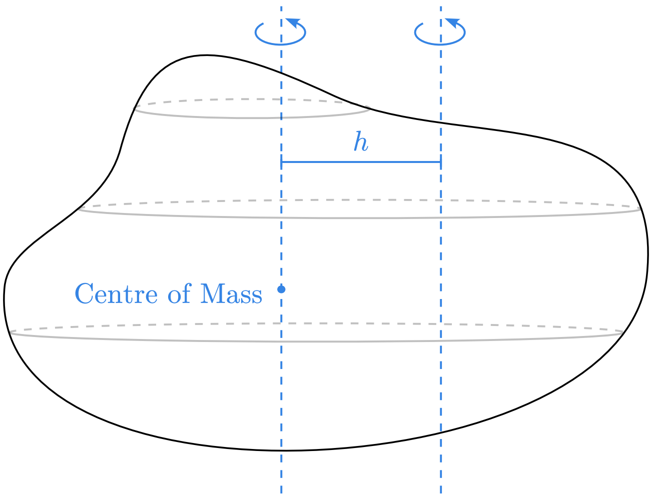
$$I_x = \int \rho y^2 \mathrm{d}^2x \quad (290)$$

$$I_y = \int \rho x^2 \mathrm{d}^2x \quad (291)$$

By inspection, we have $I_z = I_x + I_y$.

5.4 Parallel Axis Theorem

Theorem 5.4 (Parallel Axis Theorem)



Let the moment of intertia through the parallel axis (not through the center of mass) be I, and the moment of inertia through the centre of mass be I_{CoM}. We have

$$I = I_{\mathrm{CoM}} + Mh^2 \quad (292)$$

where M is the total mass of the body, and h is the distance between the two axes.

Proof. To prove this, we will express all positions relative to the centre of mass. Choose an origin on the parallel axis, and let $\mathbf{x}_i$ be the position of particle i relative to this origin. Then

$$\mathbf{x}_i = \mathbf{R} + \mathbf{y}_i \quad \cdot 293$$

where $\mathbf{R}$ is the position of the centre of mass, and $\mathbf{y}_i$ is the position of particle i relative to the centre of mass. Note

$$\sum_i m_i \mathbf{y}_i = \mathbf{0}. \quad \cdot 294$$

Lecture 12 · 2026-02-19

We have

$$I = \sum_i m_i \underbrace{(\hat{\mathbf{n}} \times \mathbf{x}_i)^2}_{d_i^2} \quad \cdot 295$$

$$= \sum_i m_i [\hat{\mathbf{n}} \times [\mathbf{R} + \mathbf{y}_i]]^2 \quad \cdot 296$$

$$= \sum_i m_i \left[(\hat{\mathbf{n}} \times \mathbf{R})^2 + 2(\hat{\mathbf{n}} \times \mathbf{R}) \cdot (\hat{n} \times \mathbf{y}_i) + (\hat{\mathbf{n}} \times \mathbf{y}_i)^2 \right] \quad \cdot 297$$

Since $\sum_i m_i \mathbf{y}_i = \mathbf{0}$, the middle term vanishes, and we have

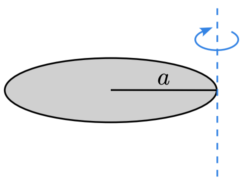
$$I = Mh^2 + I_{\text{CoM}}, \quad \cdot 298$$

by noting that $h = |\hat{\mathbf{n}} \times \mathbf{R}|$ and $I_{\text{CoM}} = \sum_i m_i (\hat{\mathbf{n}} \times \mathbf{y}_i)^2$.

Remark. This theorem implies that I_{CoM} is lower than I about any parallel axis.

Example 5.5

Consider a rotating disc about an axis through its edge.



We have

$$I = I_{\text{CoM}} + Ma^2 = \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2. \quad \cdot 299$$

5.5 Motion of Rigid Bodies

We will now consider the cases where CoM moves as the body rotates. We have

$$\mathbf{x}_i = \mathbf{R}(t) + \mathbf{y}_i \quad \cdot 300$$

where the $\mathbf{y}_i$ term will capture the rotation about the CoM if

$$\dot{\mathbf{y}} = \boldsymbol{\omega} + \mathbf{y}_i. \quad \cdot 301$$

The velocity of the body is

$$\dot{\mathbf{x}} = \dot{\mathbf{R}} + \dot{\mathbf{y}}_i. \tag{302}$$

For the kinetic energy, we have shown in Eq · 232 that

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_i m_i \dot{\mathbf{y}}_i{}^2 \tag{303}$$

$$= \underbrace{\frac{1}{2}M\dot{\mathbf{R}}^2}_{\text{translational kinetic energy}} + \underbrace{\frac{1}{2}I_{\text{CoM}}\omega^2}_{\text{rotational kinetic energy}}. \tag{304}$$

Then the full energy of the body is $E = T + V$, where we have previously shown that

$$V = \sum_i V_i(\mathbf{x}_i) + \underbrace{\sum_{i<j} V_{ij}\left(\left|\mathbf{x}_i - \mathbf{x}_j\right|\right)}_{\text{constant for rigid body so drops out from Newton's equation}}. \tag{305}$$

Consider a nice case, where

$$V_i(x_i) = m_i g z_i, \tag{306}$$

then,

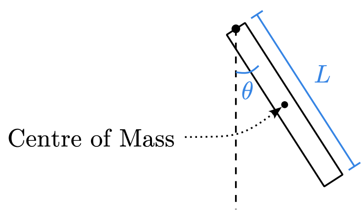
$$V = \sum_i V_i = g\sum_i m_i z_i = gMR_z \tag{307}$$

where R_z is the z-component of the centre of mass. Hence a rigid body is just like a point particle with mass M, located at the centre of mass.

> ***Remark.*** In some cases, it may be easiest to consider axes that do no pass through the CoM.

Example 5.6

Consider a rigid rod pendulum of mass M.



We will carry out calculation in two ways:

1. the end point (pivot) is fixed, so there is only rotational motion about this point, giving

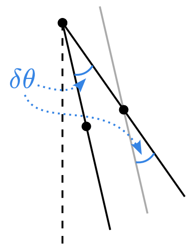
$$T = \frac{1}{2}I\dot{\theta}^2 \tag{308}$$

in which case $\omega = \dot{\theta}$ and $I = \frac{1}{3}ML^2$.

2. the CoM is moving with $V_{\text{CoM}} = \frac{L}{2}\dot{\theta}$. Hence

$$T = \frac{1}{2}MV_{\mathrm{CoM}}^2 + \frac{1}{2}I_{\mathrm{CoM}}\dot{\theta}^2. \quad \cdot 309$$

To check that the angular velocity about CoM is the same as that about the pivot, we have



Hence,

$$T = \frac{1}{2}M\left(\frac{L}{2}\right)^2\dot{\theta}^2 + \frac{1}{2}\left[I - M\left(\frac{L}{2}\right)^2\right]\dot{\theta}^2 = \frac{1}{2}I\dot{\theta}^2. \quad \cdot 310$$

To understand the motion, consider the energy

$$E = \frac{1}{2}I\dot{\theta}^2 - Mg\left(\frac{L}{2}\right)\cos\theta. \quad \cdot 311$$

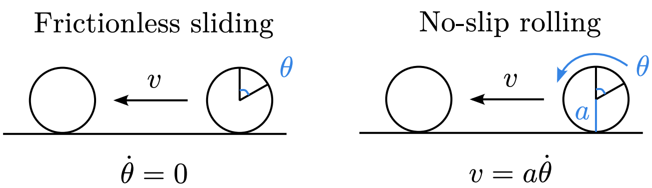
Imposing $\dot{E} = 0$ gives

$$\frac{1}{2}I \cdot 2\dot{\theta}\ddot{\theta} + Mg\left(\frac{L}{2}\right)\sin\theta \cdot \dot{\theta} = 0 \quad \cdot 312$$

$$I\ddot{\theta} = -Mg\left(\frac{L}{2}\right)\sin\theta. \quad \cdot 313$$

Example 5.7 (Rolling Ball)

No-slip rolling occurs when the friction between the ball and the ground is so strong that the relative velocity between the point of contact and the ground is zero. Compare:



The kinetic energy is given by, with $\omega = \dot{\theta}$ and noting that $\dot{\theta} = \frac{v}{a}$,

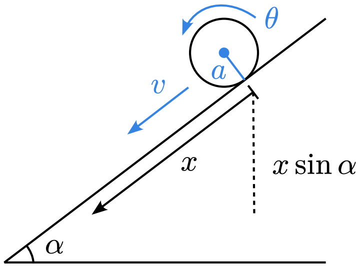
$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \quad \cdot 314$$

$$= \frac{1}{2}\underbrace{\left(M + \frac{I}{a^2}\right)}_{\text{effective mass}}v^2 \quad \cdot 315$$

Important. Because there is no relative velocity between the point of contact and the ground, no work is done by the friction force, so the energy is conserved.

The only role of rolling is to impose the no-slip condition.

In the case where the ball rolls down a slope,



The conserved energy is

$$E = \frac{1}{2}\left(M + \frac{I}{a^2}\right)\dot{x}^2 - Mgx\sin\alpha. \quad \cdot 316$$

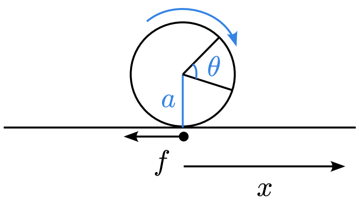
By imposing $\dot{E} = 0$, we have

$$\left(M + \frac{I}{a^2}\right)\ddot{x} = Mg\sin\alpha. \quad \cdot 317$$

Lecture 13 · 2026-02-21

Now consider a ball rolling on a horizontal surface, where we will demonstrate the conservation of energy. We have

$$E = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 \quad \cdot 318$$



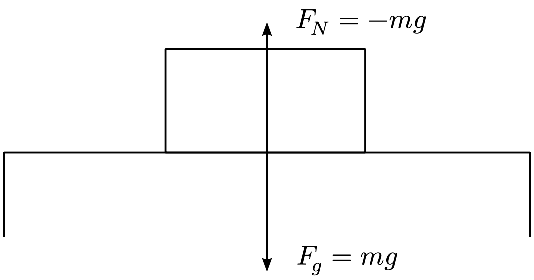
Then

$$\begin{aligned} \frac{dE}{dt} &= M\dot{x}\ddot{x} + I\dot{\theta}\ddot{\theta} && \cdot 319 \\ &= \dot{x}(-f) + \dot{\theta}(af) && \cdot 320 \\ &= f\left(-\dot{x} + a\dot{\theta}\right) && \cdot 321 \\ &= 0 && \cdot 322 \end{aligned}$$

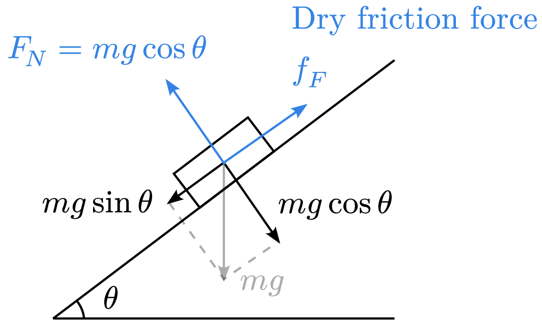
where f is the friction force, and we have used the no-slip condition $\dot{x} = a\dot{\theta}$.

5.6 Normal Forces

Objects on a table does not fall through the table, because the table exerts a **normal force** on the object that pushes it away. [Microscopic origin of normal forces is electrostatic repulsion and the Pauli exclusion principle.]



At an angle,



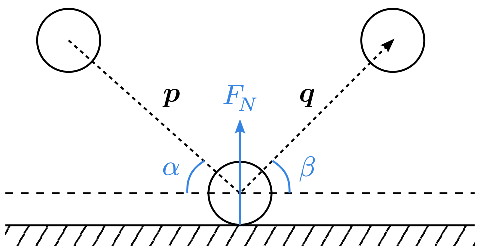
Normal force does not prevent the object from sliding. Sliding is prevented by the dry friction force f_F. Once the object starts moving, we typically have

$$f_F = \mu F_N \tag{323}$$

where μ is the coefficient of friction.

> ***Remark.*** f_F is independent of speed, but always opposes the direction of motion.

Normal forces also product elastic bounces:



Normal force on impact is in the direction normal to the surface at point of impact.

Definition 5.8 (Elastic Collision)

An **elastic collision** is a collision where energy is conserved.

Note that by conservation of total momentum,

$$p_y = q_y + Q_y \tag{324}$$

where Q_y is the momentum of the Earth or the wall after the collision.

By conservavtion of energy,

$$\frac{p^2y}{2m} = \frac{qy^2}{2m} + \frac{Q_y^2}{2M} \quad \cdot 325$$

where m is the mass of the ball, and M is the mass of the Earth or the wall. Clearly $M \gg m$.

Substituting Eq · 324 into Eq · 325 gives

$$\frac{1}{2m}\left(q_y^2 + 2q_yQ_y + Q_y^2\right) = \frac{qy^2}{2m} + \frac{Q_y^2}{2M} \quad \cdot 326$$

$$\frac{1}{2m}\left(2q_yQ_y + Q_y^2\right) = \frac{Q_y^2}{2M} \quad \cdot 327$$

If $M \gg m$, then $\frac{Q_y^2}{2M} \approx 0$, and we have

$$\frac{Q_y}{2m}\left(2q_y + Q_y\right) \approx 0 \quad \cdot 328$$

$$Q_y \approx -2q_y \quad \cdot 329$$

$$p_y \approx -q_y. \quad \cdot 330$$

Hence we can conclude that

$$\beta = \alpha. \quad \cdot 331$$

The change Δp in momentum over a short time is called an **impulse** I.

6 Rotating Reference Frames

Rotating reference frames (RRFs) are important examples of non-inertial frames.

6.1 Newton's Equations in a Rotating Reference Frame

An inertial frame S has Cartesian axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, and a rotating frame S' has axes $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$.

From the perspective of the inertial frame, the $\boldsymbol{e}'_i$ axes rotates with angular velocity $\boldsymbol{\omega}$.

$$\dot{\boldsymbol{e}}'_i = \boldsymbol{\omega} \times \boldsymbol{e}'_i. \tag{332}$$

In the two frames, the position of a particle is, repsectively,

$$\boldsymbol{x} = x_i \boldsymbol{e}_i = x'_i \boldsymbol{e}'_i. \tag{333}$$

We wish to find $\ddot{\boldsymbol{e}}'_i$ in terms of $\boldsymbol{\omega}$ and $\boldsymbol{e}'_i$. We have

$$\begin{aligned}
\dot{\boldsymbol{x}} = \underbrace{\dot{x}_i \boldsymbol{e}_i}_{(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t})_S} &= \dot{x}'_i \boldsymbol{e}'_i + x'_i \dot{\boldsymbol{e}}'_i && (334)\\
&= \dot{x}'_i \boldsymbol{e}'_i + x'_i \boldsymbol{\omega} \times \boldsymbol{e}'_i && (335)\\
&= \underbrace{\dot{x}'_i \boldsymbol{e}'_i}_{(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t})_{S'}} + \boldsymbol{\omega} \times \boldsymbol{x} && (336)
\end{aligned}$$

$$\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \boldsymbol{x}. \tag{337}$$

where $\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_S$ means the derivatives of components of $\boldsymbol{x}$ with respect to t in the frame S.

The difference between the two time derivatives is just the relative velocity of the two frames.

For Newton's second law, we need to find the acceleration,

$$\begin{aligned}
\ddot{\boldsymbol{x}} &= \ddot{x}_i \boldsymbol{e}_i && (338)\\
&= \ddot{x}'_i \boldsymbol{e}'_i + \underbrace{\dot{x}'_i \dot{\boldsymbol{e}}'_i}_{\substack{= \dot{x}'_i \boldsymbol{\omega} \times \boldsymbol{e}'_i \\ = \boldsymbol{\omega} \times (\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t})_{S'}}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \underbrace{\boldsymbol{\omega} \times \dot{\boldsymbol{x}}}_{= \boldsymbol{\omega} \times (\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t})_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x})} && (339)
\end{aligned}$$

i.e.

$$\left(\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}\right)_S = \left(\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_{S'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}). \tag{340}$$

In the inertial frame, we have

$$m\left(\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}\right)_S = \boldsymbol{F}. \tag{341}$$

Hence,

$$m\left(\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}\right)_{S'} = \boldsymbol{F} \underbrace{- \underbrace{m\dot{\boldsymbol{\omega}} \times \boldsymbol{x}}_{\text{Euler force}} - \underbrace{2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_{S'}}_{\text{Coriolis force}} - \underbrace{m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x})}_{\text{Centrifugal force}}}_{\text{Fictitious forces}}. \tag{342}$$

A free particle does not move in a straight line in the rotating frame.

Lecture 14 · 2026-02-24

Consider the rotating frame of the earth. We have

$$\omega_{\text{rot}} = 2\frac{\pi}{1\text{ day}} \approx 7 \cdot 10^{-5}\text{ s} \tag{343}$$

$$R_{\text{Earth}} \approx 6 \cdot 10^{3}\text{ km} \tag{344}$$

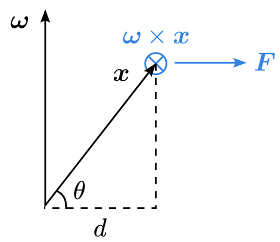
We shall neglect the small wobbling of the Earth, so assume $\dot{\boldsymbol{\omega}} = \mathbf{0}$, and hence no Euler force.

6.2 Centrifugal Force

We have

$$\boldsymbol{F}_{\text{cent}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}). \tag{345}$$

It points away from the axis of rotation, as shown in the following diagram.



For the size of the force,

$$|\boldsymbol{F}_{\text{cent}}| = m\omega^2 r \cos\theta. \tag{346}$$

The centrifugal force is conservative, with

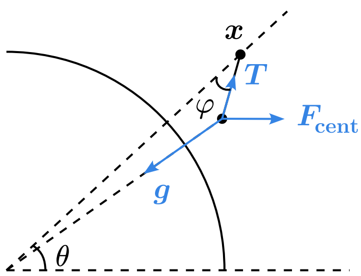
$$\boldsymbol{F}_{\text{cent}} = -\boldsymbol{\nabla} V_{\text{cent}} \tag{347}$$

$$V_{\text{cent}} = -\frac{m}{2}|\boldsymbol{\omega} \times \mathbf{x}|^2 = -\frac{m}{2}\omega^2 r^2 \cos^2\theta. \tag{348}$$

Hence, potential energy is lowered by moving away from the axis of rotation.

Example 6.1 (Hanging String)

Consider a hanging string on the Earth.



Rather than hanging vertically downwards, the pendulum hangs at an angle φ to the vertical. We wish to find φ.

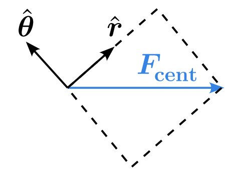
The forces acting on the particle satisfy

$$\boldsymbol{g} = -mg\hat{\boldsymbol{r}}. \quad \cdot 349$$

[The string is short compared to R_{Earth}, so it does not matter whether we use $\hat{r}$ at the top or the bottom of the string.]

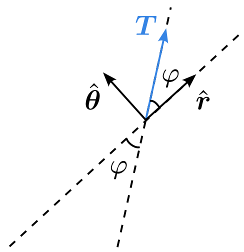
$$\boldsymbol{F}_{\text{cent}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}) \quad \cdot 350$$

$$= m\omega^2 r \cos\theta\left(\cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right). \quad \cdot 351$$



To hold the string together, there must be a force exerted by the molecules on the string that balances the other forces, which is the tension.

$$\boldsymbol{T} = T\cos\varphi\hat{\boldsymbol{r}} + T\sin\varphi\hat{\boldsymbol{\theta}}. \quad \cdot 352$$



The net force on the particle is zero, so

$$\boldsymbol{g} + \boldsymbol{F}_{\text{cent}} + \boldsymbol{T} = \boldsymbol{0}. \quad \cdot 353$$

We have 2 equations (for $\hat{r}$ and $\hat{\theta}$) and 2 unknowns (T and φ), so we can solve for φ:

$$\tan\varphi = \frac{\omega^2 R\cos\theta\sin\theta}{g - \omega^2 R\cos^2\theta}. \quad \cdot 354$$

At the equator ($\theta = 0$), the gravity is a bit weaker, but $\varphi = 0$.

When $\theta = 45°$, $\varphi \approx 10^{-4}$, so the effect is very small.

6.3 Coriolis Force

In Eq · 342, we have

$$\boldsymbol{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \boldsymbol{v} \quad \cdot 355$$

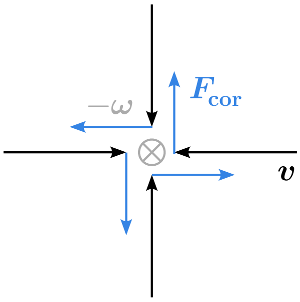
where $\boldsymbol{v}$ is the velocity of the particle in the rotating frame.

Note that this is similar to Lorentz force with $\boldsymbol{B} \to \boldsymbol{\omega}$, so moving particles will turn in circles.

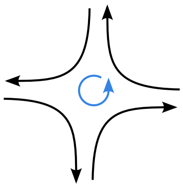
Example 6.2

Coriolis force is responsible for the formation of hurricanes.

When a low pressure region forms, air particles move in, and the Coriolis force bends them:



Each molecule of air in bent clockwise in the northern hemisphere [by the right hand rule with $-\boldsymbol{\omega}$ going into the plane], which leads to an anticlockwise swirling motion.



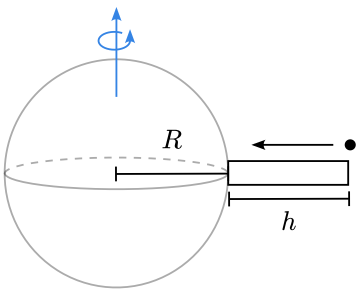
In the southern hemisphere, the Coriolis force bends particles anticlockwise, leading to a clockwise swirling motion.

Motion along the Earth's surface is not in general perpendicular to the axis of rotation $\boldsymbol{\omega}$. Hence, the effect of Coriolis force is typically weaker near the equator. There are empirical observations that hurricanes do not form within near the equator.

[$\boldsymbol{\omega} \times \mathbf{v}$ can be substantial near the equator if $\mathbf{v}$ moves along the equator, but it pushes particles vertically, and it need to compete with gravity, which is much stronger.]

Example 6.3

Consider dropping a ball from a tower on the euqator. We will consider where the ball lands.



Initially,

$$\ell = \omega(R+h)^2. \quad \cdot 356$$

As the ball falls, the distance to the axis decreases, so the angular velocity must increase to conserve angular momentum.

At the foot of the tower, $\ell = \omega' R^2$, which must give $\omega' > \omega$, and hence the ball rotates faster than the Earth, so it lands slightly east of the foot of the tower.

In the rotating frame,

$$\ddot{\boldsymbol{x}} = \boldsymbol{g} - 2\boldsymbol{\omega} \times \dot{\boldsymbol{x}}. \quad \cdot 357$$

[We can neglect the centrifugal force since it does not affect the horizontal motion.]

Integrating once gives

$$\dot{\boldsymbol{x}} = \boldsymbol{g}t - 2\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0) \quad \cdot 358$$

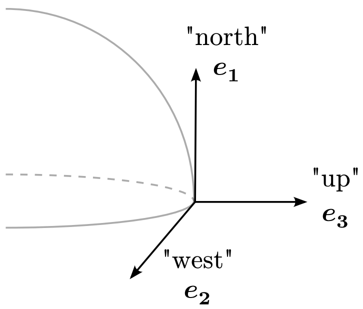
where $\boldsymbol{x}_0$ is the initial position of the ball. Subsituting Eq · 358 into Eq · 357 gives

$$\ddot{\boldsymbol{x}} = \boldsymbol{g} - 2\boldsymbol{\omega} \times \boldsymbol{g}t + \underbrace{4\boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0))}_{\text{same order as centrifugal force}}. \quad \cdot 359$$

The last term acts in the vertical direction and is small, so we can neglect it. Hence, integrating twice gives

$$\boldsymbol{x} = \boldsymbol{x}_0 + \frac{1}{2}\boldsymbol{g}t^2 - \frac{1}{3}\boldsymbol{\omega} \times \boldsymbol{g}t^3. \quad \cdot 360$$

Consider the following right-handed set of basis:



$$\boldsymbol{\omega} = \omega \boldsymbol{e}_1 \quad \cdot 361$$
$$\boldsymbol{g} = -g\boldsymbol{e}_3. \quad \cdot 362$$
$$\boldsymbol{x}_0 = (R + h)\boldsymbol{e}_3. \quad \cdot 363$$

Then, substituting back into Eq · 360 gives

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ -\frac{1}{3}\omega g t^3 \\ R + h - \frac{1}{2}gt^2 \end{pmatrix}. \quad \cdot 364$$

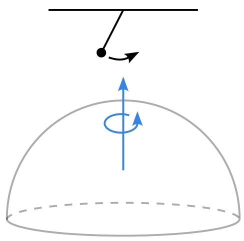
Clearly, x_2 is negative at positive t, so the ball lands slightly east of the foot of the tower.

Lecture 15 · 2026-02-26

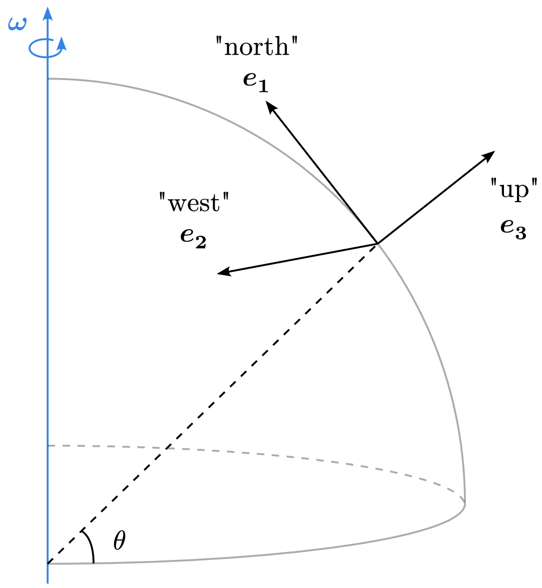
6.4 Foucault's Pendulum

Foucault's pendulum demonstrates the rotation of the Earth.

As the Earth rotates under the pendulum, from the point of view of someone on the Earth, it will look like the pendulum rotates.



At a general latitude,

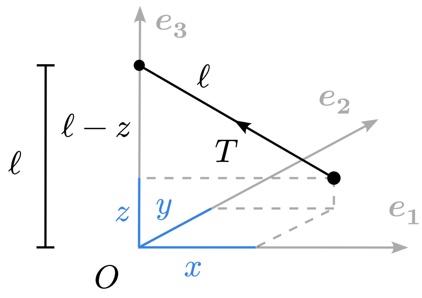


In the basis on the Earth's surface, we have

$$\boldsymbol{x} = (x, y, z) \tag{365}$$

$$\boldsymbol{g} = (0, 0, -g) \tag{366}$$

$$\boldsymbol{\omega} = (\omega \cos\theta, 0, \omega \sin\theta) \tag{367}$$



The tension in the string is

$$\boldsymbol{T} = T\left(-\frac{x}{\ell}, -\frac{y}{\ell}, \frac{\ell - z}{\ell}\right). \tag{368}$$

Since the string doesn't break, we have

$$x^2 + y^2 + (\ell - z)^2 = \ell^2. \tag{369}$$

Now, for the equations of motion,

$$m\ddot{\boldsymbol{x}} = \boldsymbol{T} + m\boldsymbol{g} - 2m\boldsymbol{\omega} \times \dot{\boldsymbol{x}}. \tag{370}$$

Note that we have all the quantities defined, with 4 equations (3 ODEs and 1 constraint) and 4 unknowns (x, y, z, and T), so we can solve for the motion of the pendulum. Our strategy is as follows

- Solve constraint for z in terms of x and y,
- Substitute into the ODEs,
- Eliminate T to get 2 ODEs for x and y,
- Solve the ODEs.

The exact solution is tedious, but the upshot is that the pendulum follows an ellipse in the x, y plane that slowly rotates, *i.e.*

$$x + \mathrm{i}\, y = \mathrm{e}^{-\mathrm{i}\,\omega t \sin\theta}\left[\alpha \cos\left(\sqrt{\frac{g}{\ell}}t\right) + \varphi \sin\left(\sqrt{\frac{g}{\ell}}t\right)\right]. \quad \cdot 371$$

The period of rotation is

$$\frac{24}{\sin\theta} \text{ hours} \approx 32 \text{ hours in Paris.} \quad \cdot 372$$

7 Special Relativity

7.1 Basic Postulates of Relativity

Maxwell's equations (1862) predicted the existence of electromagnetic waves. These travel in the vaccum at the speed of light,

$$c = 299\,792\,458\,\mathrm{m\,s^{-1}}. \quad \cdot 373$$

The quantity is exact due to the definition of the metre.

However, a theory with a preferred velocity cannot be Galilean invariant, since Galilean theories can only have relative velocities. [This could be fine, such as the case of the sound in air which travels at a definitive speed, but this is relative to the air, which is at rest. In the case of light, there is no medium, so there is no reference frame in which the light is at rest.] It was assumed that light must also be travelling in some fixed medium, called the luminiferous æther.

Michelson and Morley (1881) performed an experiment that effectively showed that if one run towards a wave of light, the speed of the light does not change. This was a strong indication that there is no such thing as the æther, and that the speed of light is invariant in all inertial frames.

Einstein (1905) postulated that there was no æther, and

1. The laws of physics are the same in all inertial frames. [This is Galileo's principle of relativity.]
2. The speed of light in vacuum is the same in all inertial frames.

7.2 Lorentz Transformations

From the postulates, we must change the rules for transforming between frames.

We shall start with one spatial dimension x.

A frame S has coordinates (x, t). A frame S' has coordinates (x', t'), and moves at a constant speed v relative to S.

According to Galileo,

$$x' = x - vt, \quad t' = t. \quad \cdot 374$$

Instead, let's allow for a general transformation,

$$x' = f(x, t), \quad t' = g(x, t). \quad \cdot 375$$

By postulate (1), law of inertia is still true, so a particle experiencing no forces moves at a constant velocity in all frames. *i.e.* for such a particle,

$$x = A + Bt \quad \text{in } S \quad \cdot 376$$
$$x' = A' + B't' \quad \text{in } S' \quad \cdot 377$$

We shall choose our frames to have a common origin *i.e.* $x = t = 0 \Leftrightarrow x' = t' = 0$. [We can always shift x and x' to achieve this.]

Hence, the transformation must map all lines in the (x, t) plane to lines in the (x', t') plane. These are precisely linear transformations. Hence, we can write

$$x' = ax + bt, \quad t' = cx + dt. \quad \cdot 378$$

Note that a, b, c, d does not depend on x or t, but can depend on v.

Lecture 16 · 2026-02-28

Consider the frame S' to be moving at speed v relative to S in S. Then, the point $x = vt$ should map to $x' = 0$. Hence,

$$x' = \gamma_v (x - vt). \quad \cdot 379$$

We will follow a few steps to determine more results about the transformation.

Step 1. $\gamma_v = \gamma_{-v}$.

Consider $\tilde{S}$ and $\tilde{S'}$ where the x-axis is inverted in direction, *i.e.*

$$\tilde{x} = -x, \quad \tilde{x'} = -x'. \quad \cdot 380$$

Since S' moves at speed v relative to S, $\tilde{S'}$ moves at speed $-v$ relative to $\tilde{S}$. Hence, the transformation from $\tilde{S}$ to $\tilde{S'}$ is given by

$$\tilde{x'} = \gamma_{-v}(\tilde{x} + vt) = -\gamma_{-v}(x - vt) \quad \cdot 381$$
$$x' = -\tilde{x'} = \gamma_{-v}(x - vt). \quad \cdot 382$$

and so $\gamma_v = \gamma_{-v}$.

Another argument for $\gamma_v = \gamma_{-v}$ is that in 3D there is no preferred direction, so γ_v can only depend on $|\mathbf{v}|$.

Step 2. We can assume that if we boost by v and then by $-v$, we should get back to the original frame, *i.e.*

$$S \underset{v}{\to} S' \underset{-v}{\to} S'' = S. \quad \cdot 383$$

i.e. the boost by $-v$ is the inverse transformation of the boost by v.

We have

$$\begin{aligned} x'' &= \gamma_{-v}(x' + vt') && \cdot 384 \\ &= \gamma_{-v}\big(\gamma_v(x - vt) + t'\big) && \cdot 385 \\ &= \gamma^2 (x - vt) + \gamma t' \qquad \text{by } \gamma_v = \gamma_{-v} && \cdot 386 \\ &= x \qquad \text{by assumption} && \cdot 387 \end{aligned}$$

So

$$t' = \gamma t + \frac{1 - \gamma^2}{\gamma v} x. \quad \cdot 388$$

Step 3. With postulate (2), the light ray $x = ct$ must map to $x' = ct'$.

We have

$$x' = ct' = c\left(\gamma t + \frac{1-\gamma^2}{\gamma v}x\right) \quad \cdot 389$$

$$= c\left(\gamma + \frac{1-\gamma^2}{\gamma}\frac{c}{v}\right)t \quad \cdot 390$$

and

$$x' = \gamma(x - vt) = \gamma(c - v)t \quad \cdot 391$$

Hence, solving for γ gives

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \quad \cdot 392$$

See that the transformation only makes sense for $v < c$. So we have derived

$$x' = \gamma(x - vt) \quad \cdot 393$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right). \quad \cdot 394$$

This is the Lorentz transformation, or the Lorentz boost. These are linear transformations, so we can invert them to get

$$x = \gamma(x' + vt') \quad \cdot 395$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \quad \cdot 396$$

i.e. $v \to -v$.

For velocities $v \ll c$, $\gamma \approx 1$ and $\frac{\gamma}{c} \to 0$, so these become the Galilean transformations *i.e.* the non-relativistic limit.

We will now explore some possibly counter-intuitive consequences of the above.

7.3 Addition of Velocities

A particle moving with speed u' in a frame S' which in turn moves at a speed v in frame S. Consider the speed u the particle moves at in frame S.

By Eq · 395 and Eq · 396, we have

$$u = \frac{x}{t} = \frac{x' + vt'}{t' + \frac{v}{c^2}x'} \quad \cdot 397$$

$$= \frac{\frac{x'}{t'} + v}{1 + \frac{v}{c^2}\frac{x'}{t'}} \quad \cdot 398$$

Therefore, we have the relativistic formula for addition of velocities:

$$u = \frac{u' + v}{1 + \frac{vu'}{c^2}}. \quad \cdot 399$$

It is easy to check that if $|u'|$ and $|v|$ are both less than c, then so is u. Hence, we cannot make an object move faster than light by adding velocities.

Example 7.1

1. Let $u' = v = \frac{c}{2}$. Then

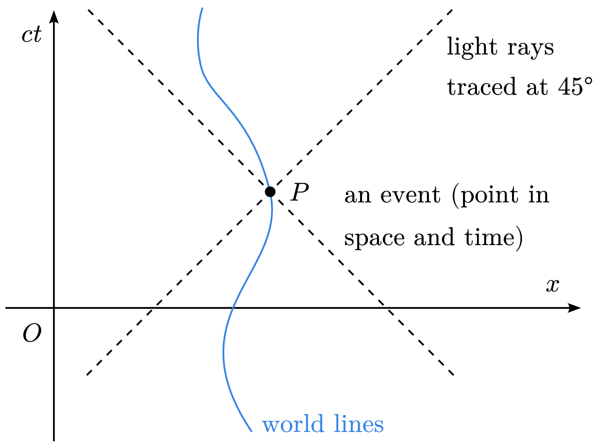
$$u = c \cdot \frac{\frac{1}{2} + \frac{1}{2}}{1 + \frac{1}{4}} = \frac{4}{5}c < c. \tag{400}$$

2. Let $u' = v = c$. Then

$$u = c \cdot \frac{1 + 1}{1 + 1} = c. \tag{401}$$

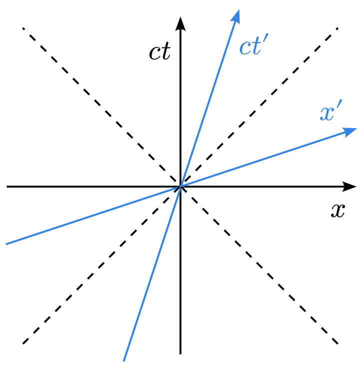
7.4 Spacetime Diagrams and Simultaneity

Consider the following type of diagram



We may put the axes for a frame S' on the spacetime diagram of S:

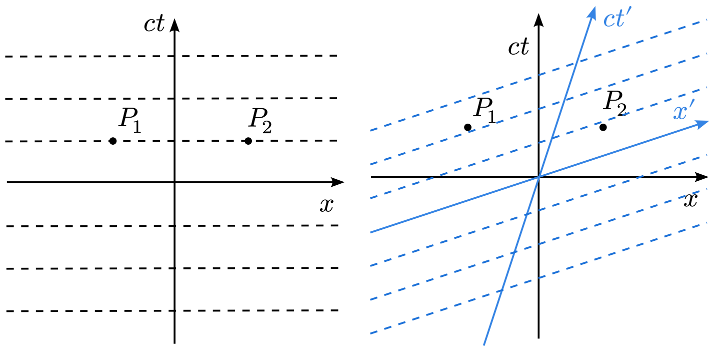
- t' axis is at $x' = 0$, which gives $ct = \frac{c}{v}x$.
- x' axis is at $t' = 0$, which gives $ct = \frac{v}{c}x$.



where the axes are symmetrical about the light ray $x = ct$.

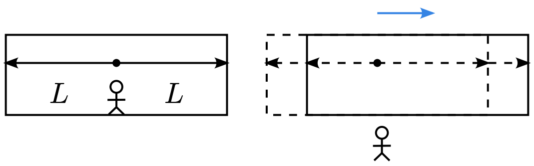
Lecture 17 · 2026-03-03

We can draw lines at constant t and t' to get



So P_1 and P_2 are simultaneous in S but not in S'. This is the relativity of simultaneity.

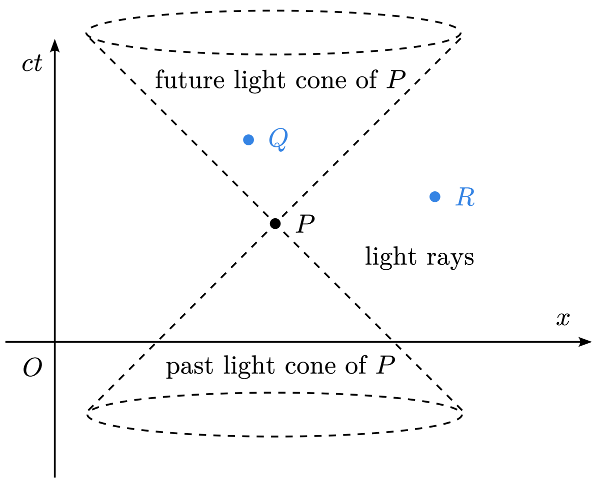
There is a direct consequence of the speed of light being the same, which is demonstrated by a famous illustration due to Einstein. Consider a train.



A light is emitted from the centre of the train. The light reaches the front and back of the train at the same time in the frame of the train. However, for an observer on the platform, the light reaches the back of the train before it reaches the front. Hence, events that are simultaneous in one frame may not be simultaneous in another frame.

[In Galilean physics, the platform observer would see light going at $c + v$ in one direction and $c - v$ in the other, reaching both ends at the same time.]

One may concern that if different frames see things at different times, we might be able to reverse cause and effect. We will see that this is not the case.



The future light cone of P consists of all points that can be influenced by P, and the past light cone of P consists of all points that can influence P.

Note that the lines of simultaneity of S' can be at most at 45° as $v \to c$. Hence, we can make R simultaneous with P, but cannot make Q simultaneous with P.

Therefore, in all frames, the future light cone of P is to the future of P, and nothing moves faster than light, so causality is ensured. *i.e.* all frames agree on what events can influence.

7.5 Time Dilation

A clock at rest in frame S' ticks at intervals T'. *i.e.* the ticks occur at

$$(0,0),(cT',0),(2cT',0),\ldots \quad \cdot 402$$

In frame S, using Eq · 395 and Eq · 396, where

$$t=\gamma\left(t'+\frac{v}{c^2}x'\right) \quad \cdot 403$$

with $x'=0$, we have the ticks occurring at

$$t=0,\gamma T',2\gamma T',\ldots \quad \cdot 404$$

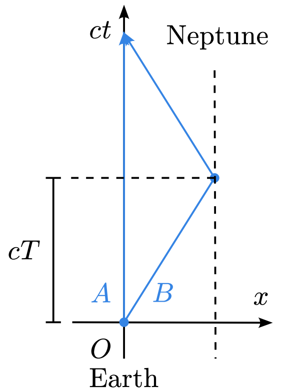
So $T=\gamma T'$. Recall that $\gamma>1$, so $T>T'$. *i.e.* a moving clock runs more slowly. [T is the time interval between ticks on the moving clock as measured in S, and T' is the time interval between ticks on the clock as measured in its rest frame S'.]

7.6 Twin Paradox [Not a Paradox]

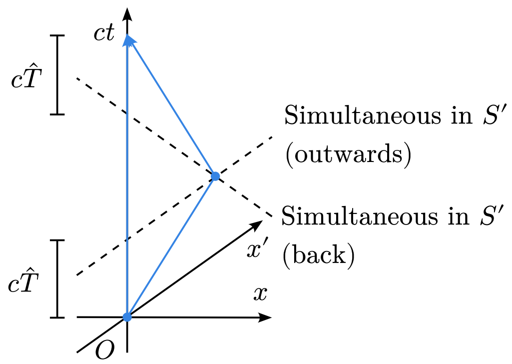
Consider twin A that stays on Earth, and B that goes to Neptune and back at almost light speed. We would investigate which one is younger.

Each twin sees the other one moving while themselves staying at rest, so each one thinks the other one is younger. This seems paradoxical.

The key asymmetry is that twin B needs to turn around to come back.



From the perspective of A, time $2T$ passes, while $2T'=\frac{2T}{\gamma}$ has passed on B's clock, so B is younger.



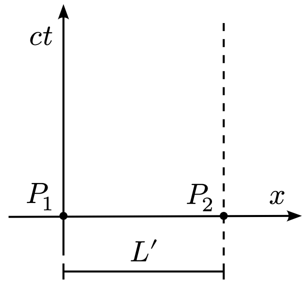
From the perspective of B, while travelling out and back, indeed A's clock runs more slowly, and $2\hat{T} = 2\frac{T'}{\gamma}$ passes in total.

However, the jump in the lines of simultaneity at the turnaround means that B sees A's clock jump forward by a large amount, so A is older.

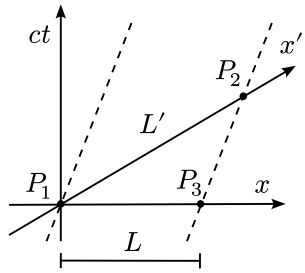
> ***Remark.*** See Example Sheet for a full treatment on this issue.

7.7 Length Contraction

A rod has length L' at rest in frame S', where the length is defined as the distance between endpoints at a fixed time.



Now consider frame S.



Lecture 18 · 2026-03-05

We would like to calculate $L = P_1P_3$ given L'.

We have

$$
\begin{aligned}
P_1 &= (0,0) \quad \text{in both frames} && \cdot\ 405 \\
P_2 &= (ct', x') = (0, L') && \cdot\ 406 \\
&= (ct, x) = \left(\gamma \frac{v}{c} L', \gamma L'\right). && \cdot\ 407 \\
P_3 \text{ has } x &= \gamma L' - v\frac{\gamma v L}{c^2} && \cdot\ 408
\end{aligned}
$$

i.e. for P_3 we have $x|_{P_3} = x|_{P_2} - vt|_{P_2}$.

Hence,

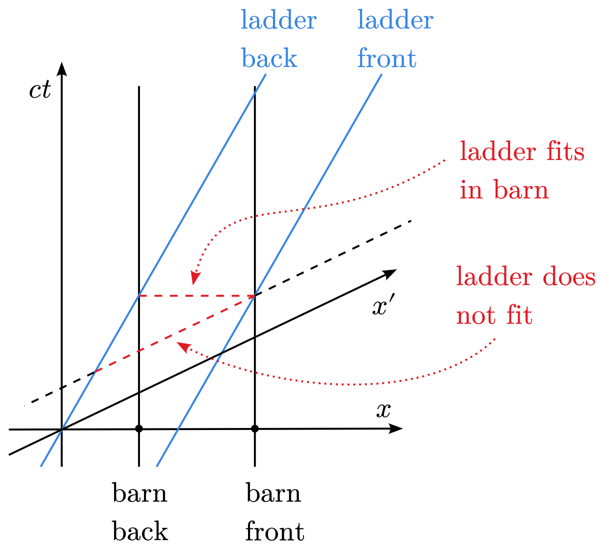
$$L = \gamma L'\left(1 - \frac{v^2}{c^2}\right) = \frac{L'}{\gamma}. \qquad \cdot\ 409$$

Therefore, moving objects appear shorter. This is called **Lorentz contraction**.

Example 7.2

Consider a ladder of length $2L$ and a barn of length L.

Due to lorentz contraction, if the ladder moves fast enough, an external observer would see the ladder fit in the barn. However, the observer on the ladder would see the barn even shorter, so the ladder cannot fit in the barn.



7.8 The Invariant Interval

Consider two events P_1 and P_2 with coordinates (ct_1, x_1) and (ct_2, x_2) in frame S. Let

$$\Delta t = t_2 - t_1 \tag{410}$$
$$\Delta x = x_2 - x_1 \tag{411}$$

each represent the separation in time and space between the two events.

The **invariant interval** is defined as

$$(\Delta s)^2 \equiv c^2(\Delta t)^2 - (\Delta x)^2. \tag{412}$$

We can check that it is indeed invariant:

$$c^2(\Delta t')^2 - (\Delta x')^2 = c^2\gamma^2\left(\Delta t - \frac{v}{c^2}\Delta x\right)^2 - \gamma^2(\Delta x - v\Delta t)^2 \tag{413}$$
$$= (\Delta t)^2\left(\gamma^2 c^2 - \gamma^2 v^2\right) + (\Delta x)^2\left(\gamma^2\frac{c^2v^2}{c^4} - \gamma^2\right) \tag{414}$$
$$= c^2(\Delta t)^2 - (\Delta x)^2. \tag{415}$$

Hence, observers may disagree about time passed and the distance between events, but they all agree on the invariant interval. We can write

$$(\Delta s)^2 = (c\Delta t \;\; \Delta x)\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{Minkowski metric}}\begin{pmatrix} c\Delta t \\ \Delta x \end{pmatrix}. \tag{416}$$

With this matrix form, the Lorentz transformation can be written as

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \quad \cdot 417$$

The invariance of $(\Delta s)^2$ is equivalent to

$$\underbrace{\begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix}}_{\text{transposed}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \cdot 418$$

i.e. Lorentz transformations preserve the Minkowski metric.

> ***Remark.*** Compare this to how rotations preserve the Euclidean metric, $\boldsymbol{R}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The Minkowski metric is not positive definite. So, we have points with

$$(\Delta s)^2 > 0 \quad \text{timelike separated} \quad \cdot 419$$

$$(\Delta s)^2 < 0 \quad \text{spacelike separated} \quad \cdot 420$$

$$(\Delta s)^2 = 0 \quad \text{lightlike separated} \quad \cdot 421$$

Note that, two points with $(\Delta s)^2 = 0$ are connected by a light ray. The Minkowski metric mesures distances in spacetime.

7.9 Rapidity

To make the analogy with rotations more clear, we will define **rapidity** φ as

$$\gamma =: \cosh \varphi. \quad \cdot 422$$

Then,

$$\sinh \varphi = \sqrt{\cosh^2 \varphi - 1} = \sqrt{\gamma^2 - 1} = \frac{\gamma v}{c}. \quad \cdot 423$$

So the Lorentz transformation can be written as

$$\begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} = \begin{pmatrix} \cosh \varphi & -\sinh \varphi \\ -\sinh \varphi & \cosh \varphi \end{pmatrix} =: \boldsymbol{\Lambda}[\varphi]. \quad \cdot 424$$

Hence, two sequential Lorentz boosts satisfy

$$\boldsymbol{\Lambda}[\varphi_1]\boldsymbol{\Lambda}[\varphi_2] = \boldsymbol{\Lambda}[\varphi_1 + \varphi_2]. \quad \cdot 425$$

i.e. rapidities add like angles in rotations.

In contrast, in terms of velocities,

$$\boldsymbol{\Lambda}(v_1)\boldsymbol{\Lambda}(v_2) = \boldsymbol{\Lambda}\left(\frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}\right), \quad \cdot 426$$

which is consistent with the relativistic addition of velocities.

7.10 Lorentz Transformations in 4 Dimensions

The 4D Minkowski metric is

$$\boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad \cdot 427$$

we write this with indices as

$$\eta_{\mu\nu} \quad \text{where} \quad \mu, \nu = 0, 1, 2, 3. \quad \cdot 428$$

An event in spacetime is given by a **4-vector**

$$\boldsymbol{X} = (ct, x, y, z) \quad \cdot 429$$

we write this with indices as

$$X^{\mu} \quad \text{where} \quad \mu = 0, 1, 2, 3 \quad \cdot 430$$

> ***Remark.*** It will be important to distinguish between lower and upper indices in Part IB.

The invariant distance between **0** and an event $\boldsymbol{X}$ is given by the inner product

$$\boldsymbol{X} \cdot \boldsymbol{X} \equiv \boldsymbol{X}^{\top} \boldsymbol{\eta} \boldsymbol{X} = X^{\mu} \eta_{\mu\nu} X^{\nu} = c^2 t^2 - x^2 - y^2 - z^2. \quad \cdot 431$$

Lecture 19 · 2026-03-07

The inner product of $\boldsymbol{X}$ is not positive definite. We call each of the cases

$$\boldsymbol{X} \cdot \boldsymbol{X} > 0 \quad \text{timelike} \quad \cdot 432$$
$$\boldsymbol{X} \cdot \boldsymbol{X} < 0 \quad \text{spacelike} \quad \cdot 433$$
$$\boldsymbol{X} \cdot \boldsymbol{X} = 0 \quad \text{lightlike or null} \quad \cdot 434$$

The 4D Lorentz transformations are 4×4 matrices $\boldsymbol{\Lambda}$ such that

$$\boldsymbol{X}' = \boldsymbol{\Lambda} \mathbf{X}. \quad \cdot 435$$

[In indices, this is $X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu}$.]

The defining feature of Lorentz transformations is that they have the inner product invariant,

$$\boldsymbol{X}' \cdot \boldsymbol{X}' = \boldsymbol{X} \cdot \boldsymbol{X} \quad \Leftrightarrow \quad \boldsymbol{\Lambda}^{\top} \boldsymbol{\eta} \boldsymbol{\Lambda} = \boldsymbol{\eta}. \quad \cdot 436$$

Consider the number of $\boldsymbol{\Lambda}$. There are 16 entries, and since both sides of the above are symmetric, we have 10 constraints. Hence, we expect to find 6 families of Lorentz transformations.

- 3 of them are rotations of the form

$$\boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{pmatrix} \quad \cdot 437$$

  which satisfy $\mathbf{R}^{\top} \mathbf{R} = \mathbf{I}$.

These give 3 independent rotations about different axes. Composition of them also includes reflections.

- 3 of them are Lorentz boosts along the 3 possible axes of the form

$$\mathbf{\Lambda} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \cdot 438$$

  and similarly for boosts along y and z axes.

These matrices $\mathbf{\Lambda}$ form the **Lorentz group**: $\mathrm{O}(1,3)$.

Because the inner product is preserved by these transformations, it automatically follows that the speed of light is the same in all frames, because null 4-vectors are null in all frames.

Subgroups with $\det \mathbf{\Lambda} = +1$ are called the **proper Lorentz group**, denoted by $\mathrm{SO}(1,3)$.

A further subgroup are those that do reserve the time direction, called the **proper orthochronous Lorentz group**, denoted by $\mathrm{SO}^+(1,3)$. For example,

$$\mathbf{\Lambda} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \cdot 439$$

is in $\mathrm{SO}(1,3)$ but not in $\mathrm{SO}^+(1,3)$.

7.11 Proper Time

We want to define a velocity that is a 4-vector. We need to find a time τ that is invariant under Lorentz transformations, and define the 4-velocity as

$$\boldsymbol{U} \coloneqq \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\tau}. \quad \cdot 440$$

Given two points along a worldline, the invariant interval $(\Delta s)^2$ is the same in all inertial frames. Hence, the **proper time** between these points is defined as

$$\Delta\tau \coloneqq \frac{\Delta s}{c}. \quad \cdot 441$$

Note that worldlines are always timelike, so $\Delta\tau$ is real.

All frames agree on $\Delta\tau$, and they can parameterise the worldline by

$$\boldsymbol{x}(\tau) \quad \text{and} \quad t(\tau). \quad \cdot 442$$

[This is a Lorentzian version of the arc length.]

Along a small segment of the worldline,

$$\mathrm{d}\tau = \sqrt{(\mathrm{d}t)^2 - \frac{\mathrm{d}\boldsymbol{x}^2}{c^2}} = \mathrm{d}t\sqrt{1 - \frac{1}{c^2}\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)^2} \quad \cdot 443$$

We can define the 3-velocity as

$$\boldsymbol{u} := \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \qquad \cdot 444$$

hence

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}} = \frac{1}{\gamma}\,\mathrm{d}t. \qquad \cdot 445$$

where here γ is a function of the instantaneous 3-velocity $\boldsymbol{u}$. Hence by above,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma. \qquad \cdot 446$$

A clock following the worldline has

$$\mathrm{d}\boldsymbol{x}' = \boldsymbol{0} \Rightarrow \mathrm{d}\tau = \mathrm{d}t'. \qquad \cdot 447$$

i.e. the proper time is the time measured by an observer following the worldline.

7.12 4-Velocity

Because τ is invariant and

$$\boldsymbol{X}(\tau) = \begin{pmatrix} ct(\tau) \\ \boldsymbol{x}(\tau) \end{pmatrix} \qquad \cdot 448$$

transforms by Lorentz: $\boldsymbol{X}' = \boldsymbol{\Lambda X}$, then

$$\boldsymbol{U} := \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\tau} = \begin{pmatrix} c\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} \end{pmatrix} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\begin{pmatrix} c \\ \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \end{pmatrix} = \gamma\begin{pmatrix} c \\ \boldsymbol{u} \end{pmatrix}. \qquad \cdot 449$$

This also transforms as

$$\boldsymbol{U}' = \boldsymbol{\Lambda U}. \qquad \cdot 450$$

The definition of a 4-vector implies that it transforms this way.

In particular, because of Eq · 450,

$$\boldsymbol{U}' \cdot \boldsymbol{U}' = \boldsymbol{U} \cdot \boldsymbol{U}. \qquad \cdot 451$$

In fact,

$$\boldsymbol{U} \cdot \boldsymbol{U} = \gamma^2 c^2 - \gamma^2 \boldsymbol{u}^2 = \left(c^2 - \boldsymbol{u}^2\right)\gamma^2 = c^2. \qquad \cdot 452$$

The relativistic 4-vector incorporates the familiar 3-velocity $\boldsymbol{u}$ into a nice Lorentzian object.

Lecture 20 · 2026-03-10

7.13 4-Momentum

The **4-momentum** is defined as

$$\boldsymbol{P} := m\boldsymbol{U} = \begin{pmatrix} m\gamma c \\ m\gamma \boldsymbol{u} \end{pmatrix}. \quad \cdot 453$$

where m is the property of the particle called **rest mass**.

The **relativistic energy** E and the **relativistic 3-momentum** $\boldsymbol{p}$ are defined by

$$E := m\gamma c^2, \quad \boldsymbol{p} := m\gamma \boldsymbol{u}. \quad \cdot 454$$

So the 4-momentum can be written as

$$\boldsymbol{P} = \begin{pmatrix} \frac{E}{c} \\ \boldsymbol{p} \end{pmatrix}. \quad \cdot 455$$

Note that

$$\boldsymbol{P}' = \boldsymbol{\Lambda}\boldsymbol{P}. \quad \cdot 456$$

i.e. the 4-momentum combines energy and momentum, analogously to how $\boldsymbol{X}$ combines time and space.

In the absense of forces, $\boldsymbol{P}$ is conserved,

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}\tau} = \boldsymbol{0}. \quad \cdot 457$$

Eq · 457 is a Lorentz-invariant generalisation of Newton's First Law. It also combines conservation of energy and momentum.

From $\boldsymbol{U} \cdot \boldsymbol{U} = c^2$, we get

$$\boldsymbol{P} \cdot \boldsymbol{P} = m^2c^2 \quad \cdot 458$$

$$\Rightarrow \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2 \quad \cdot 459$$

$$\Rightarrow E^2 = \boldsymbol{p}^2c^2 + m^2c^4. \quad \cdot 460$$

In the non-relativistic limit, $\frac{|\boldsymbol{u}|}{c} \ll 1$, we get

$$\boldsymbol{p} \approx m\boldsymbol{u} \quad \cdot 461$$

$$E = m\gamma c^2 \quad \cdot 462$$

$$= \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \quad \cdot 463$$

$$= mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots\right) \quad \cdot 464$$

$$\approx \underbrace{mc^2}_{\text{rest mass energy}} + \underbrace{\frac{1}{2}mu^2}_{\text{non-relativistic kinetic energy}}. \quad \cdot 465$$

The rest mass energy term is a new consequence fo relativity, where we can predict that mass leads to an energy.

Now consider the high velocity limit

$$\boldsymbol{p} = \underbrace{\gamma m}_{\substack{\text{relativistic}\\ \text{mass}}} \boldsymbol{u} \qquad \cdot 466$$

where as $u \to c$, $\gamma \to \infty$. So $\gamma m \to \infty$.

i.e. more and more forces are needed to accelerate the particle. In particular, any finite force cannot accelerate a particle to the speed of light.

Hence, a particle with $m \neq 0$ (called a massive particle) has $u < c$.

7.14 Massless Particle

In Galilean physics, the notion of a massless particle does not make sense. Our relativistic expression

$$\boldsymbol{P} \cdot \boldsymbol{P} = m^2 c^2 \qquad \cdot 467$$

suggests that massless particles should have

$$\boldsymbol{P} \cdot \boldsymbol{P} = 0. \qquad \cdot 468$$

This imply that the 4-momentum of a massless particle lies along a light ray.

4-velocity does not exist for a massless particle, and the 4-momentum is the more fundamental concept.

Using Eq · 431 and Eq · 468, we have

$$\boldsymbol{P} = \frac{E}{c}\begin{pmatrix} 1 \\ \hat{\boldsymbol{n}} \end{pmatrix} \qquad \cdot 469$$

where $\hat{\boldsymbol{n}}$ satisfies $\hat{\boldsymbol{n}}^2 = 1$. We can interpret this as

$$\hat{\boldsymbol{n}} := \frac{\boldsymbol{u}}{c} \qquad \cdot 470$$

where $\boldsymbol{u}$ is the 3-velocity, so that $\boldsymbol{u}^2 = c^2$, hence massless particles moves at the speed of light.

i.e.

$$m \neq 0 \quad \Rightarrow \quad |\boldsymbol{u}| < c \qquad \cdot 471$$
$$m = 0 \quad \Rightarrow \quad |\boldsymbol{u}| = c. \qquad \cdot 472$$

along a light ray $\mathrm{d}\tau = 0$, since any points on the trajectory are lightline separated.

Hence no time passes for a massless particle.

There are only two known massless particles: photons and gravitons. We only conider photons (the particles of light) in this course.

From quantum mechanics, we have for a photon,

$$E = \hbar\omega = \hbar\frac{2\pi c}{\lambda} \qquad \cdot 473$$

where ω is the angular frequency of the photon, and λ is the wavelength of the photon.

Length contraction means that different observers see different wavelengths of light, and hence sees different energies. (See Example Sheet.)

7.15 Particle Physics

Particle accelerators like CERN work by colliding particles at relativistic speeds. The proper framework for this is quantum field theory, but nonetheless we can learn some things from

$$\boldsymbol{P} = \text{constant}. \quad \cdot 474$$

i.e. the 4-momentum $\boldsymbol{P}$ must be the same before and after the collision.

Basic processes include particle delay and particle collisions.

7.15.1 Particle Decay

Heavy particles are often unstable and decay into lighter particles, *i.e.* the Higgs boson decays in $\sim 10^{-22}$ seconds.

Such particles are detected by their decay products, for example,

$$h \to \gamma\gamma \quad \cdot 475$$

where h is the Higgs boson and γ is a photon.

From Large Hadron Collider,

$$m_h c^2 \approx 125\,\text{GeV}. \quad \cdot 476$$

which is about $\sim 10^5$ times heavier than an electron.

By conservation of 4-momentum,

$$\boldsymbol{P}_h = \boldsymbol{P}_\gamma + \boldsymbol{P}_{\gamma'}. \quad \cdot 477$$

Before solving this, we need to choose a frame to work in. The two canonical options are

1. the “lab” frame, in which the particles are moving.
2. the “centre of mass” (or “centre of momentum”) frame, in which the total 3-momentum is zero.

(2) is often more convenient. For decays, (2) is the rest frame of the unstable particle.

Lecture 21 · 2026-03-12

In the CoM frame, we have

$$\boldsymbol{P}_h = \begin{pmatrix} m_h c \\ \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{P}_\gamma = \frac{E_\gamma}{c}\begin{pmatrix} 1 \\ \hat{\boldsymbol{n}} \end{pmatrix}, \quad \boldsymbol{P}_{\gamma'} = \frac{E_{\gamma'}}{c}\begin{pmatrix} 1 \\ \hat{\boldsymbol{n}}' \end{pmatrix}. \quad \cdot 478$$

By conservation of 4-momentum, we have

$$m_h c = \frac{E_\gamma}{c} + \frac{E_{\gamma'}}{c}, \quad \frac{E_\gamma}{c}\hat{\boldsymbol{n}} + \frac{E_{\gamma'}}{c}\hat{\boldsymbol{n}}' = \boldsymbol{0}. \quad \cdot 479$$

Hence,

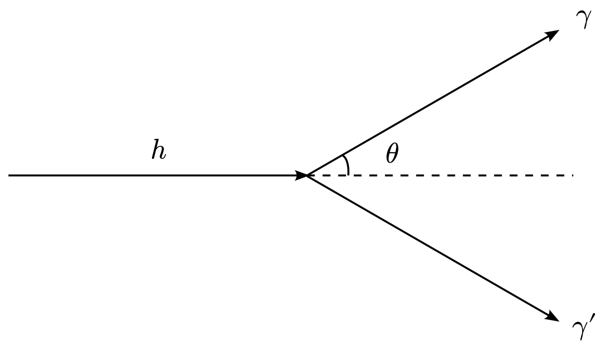
$$\begin{cases} E_\gamma = E_{\gamma'} \\ \hat{\boldsymbol{n}} = -\hat{\boldsymbol{n}}'. \end{cases} \quad \cdot 480$$

i.e. the two photons emerge with opposite 3-momenta and equal energies. Also,

$$E_\gamma = E_{\gamma'} = \frac{1}{2} m_h c^2. \qquad \cdot 481$$

Hence, each photon carries half of the rest mass energy of the Higgs boson. *i.e.* rest mass energy has been converted into kinetic energy.

In the lab frame,



consider the angle θ. The idea is to use conservation $\boldsymbol{P_h} = \boldsymbol{P_\gamma} + \boldsymbol{P_{\gamma'}}$ and invariants [quantities that do not change under Lorentz tranformations, such as the inner product of 4-vectors]. Since $\boldsymbol{P} \cdot \boldsymbol{P} = 0$ for photons, we have

$$\boldsymbol{P_h} - \boldsymbol{P_\gamma} = \boldsymbol{P_{\gamma'}} \Rightarrow \boldsymbol{P_{\gamma'}} \cdot \boldsymbol{P_{\gamma'}} = \left(\boldsymbol{P_h} - \boldsymbol{P_\gamma}\right) \cdot \left(\boldsymbol{P_h} - \boldsymbol{P_\gamma}\right) \qquad \cdot 482$$

$$\boldsymbol{0} = \boldsymbol{P_h^2} - 2\boldsymbol{P_h} \cdot \boldsymbol{P_\gamma} + \boldsymbol{P_\gamma^2} \qquad \cdot 483$$

$$= m_h^2 c^2 - 2\boldsymbol{P_h} \cdot \boldsymbol{P_\gamma} \qquad \cdot 484$$

$$2\boldsymbol{P_h} \cdot \boldsymbol{P_\gamma} = m_h^2 c^2. \qquad \cdot 485$$

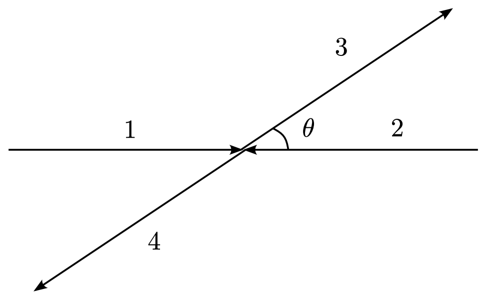
Now, since $\boldsymbol{P_h} = \binom{m_h c}{\boldsymbol{0}}$ and $\boldsymbol{P_\gamma} = \frac{E_\gamma}{c}\binom{1}{\hat{n}}$, we have

$$m_h^2 c^2 = 2\left(\frac{E_h E_\gamma}{c^2} - \frac{E_\gamma}{c}\left|\boldsymbol{p_h}\right| \cos\theta\right) \qquad \cdot 486$$

Using this alongside Eq · 460, we can solve for θ given E_h and E_γ.

7.15.2 Particle Collisions

Consider the process of two particles of mass m colliding, in the centre of mass frame, scattering at an angle θ,



By conservation of 4-momentum, we have

$$\boldsymbol{P_1} + \boldsymbol{P_2} = \boldsymbol{P_3} + \boldsymbol{P_4}. \qquad \cdot 487$$

In the CoM frame, we have

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{0} = \boldsymbol{p}_3 + \boldsymbol{p}_4. \quad \cdot 488$$

Hence $|\boldsymbol{p}_1| = |\boldsymbol{p}_2|$ and $|\boldsymbol{p}_3| = |\boldsymbol{p}_4|$. Therefore,

$$E_1 = E_2, \quad E_3 = E_4. \quad \cdot 489$$

Considering the time component of Eq · 487, we have

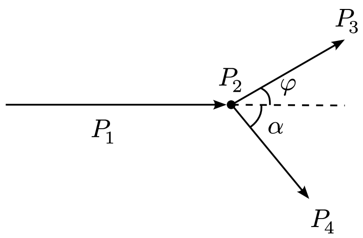
$$E_1 = E_2 = E_3 = E_4 \quad \cdot 490$$
$$\Rightarrow \quad |\boldsymbol{p}_1| = |\boldsymbol{p}_2| = |\boldsymbol{p}_3| = |\boldsymbol{p}_4| \quad \cdot 491$$

i.e. any angle θ is allowed, but momenta and energy afterwards equal the initial momenta and energy. All momenta are in the same plane, so we can pick, for example,

$$\boldsymbol{p}_1 = |\boldsymbol{p}_1| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -\boldsymbol{p}_2 \quad \cdot 492$$

$$\boldsymbol{p}_3 = |\boldsymbol{p}_3| \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = -\boldsymbol{p}_4. \quad \cdot 493$$

In the lab frame,



where P_2 is at rest, hit by P_1. Our objective is to find the angle φ.

The important step is to use invariants. We firstly have

$$\boldsymbol{P}_1 + \boldsymbol{P}_2 = \boldsymbol{P}_3 + \boldsymbol{P}_4. \quad \cdot 494$$

Since we are looking for information related to $\boldsymbol{P}_3$, the best way to eliminate $\boldsymbol{P}_4$ is to take the inner product with itself, so that

$$(\boldsymbol{P}_1 + \boldsymbol{P}_2 - \boldsymbol{P}_3)^2 = \boldsymbol{P}_4{}^2 = m^2c^2 \quad \cdot 495$$

$$m^2c^2 = \underbrace{\boldsymbol{P}_1{}^2 + \boldsymbol{P}_2{}^2 + \boldsymbol{P}_3{}^2}_{3m^2c^2} + 2\boldsymbol{P}_1 \cdot \boldsymbol{P}_2 - 2\boldsymbol{P}_1 \cdot \boldsymbol{P}_3 - 2\boldsymbol{P}_2 \cdot \boldsymbol{P}_3. \quad \cdot 496$$

Since $\boldsymbol{P}_2$ is at rest,

$$\boldsymbol{P}_2 = \begin{pmatrix} mc \\ \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{P}_1 = \begin{pmatrix} \frac{E_1}{c} \\ \boldsymbol{p}_1 \end{pmatrix}, \quad \boldsymbol{P}_3 = \begin{pmatrix} \frac{E_3}{c} \\ \boldsymbol{p}_3 \end{pmatrix}. \quad \cdot 497$$

Hence,

$$2m^2c^2 + 2m(E_1 - E_3) - \frac{2E_1E_3}{c^2} = -2\,|\boldsymbol{p}_1|\,|\boldsymbol{p}_3| \cos\varphi. \quad \cdot 498$$

We have obtained $\cos\theta$ in terms of E_1, E_3 and $\boldsymbol{p}_1, \boldsymbol{p}_3$. Using Eq · 460 eliminates the $\boldsymbol{p}_i$ terms.

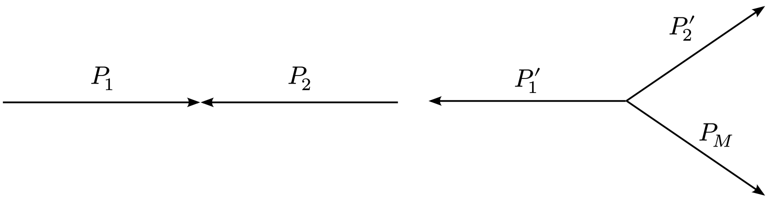
See Example Sheet 4 for Compton scattering, which describes a massless particle scattering off a massive one.

Lecture 22 · 2026-03-14

7.15.3 Particle Creation

If two particles collide with enough energy, some of that energy can be used to create a third particle. This is usually how we discover new particles.

In the CoM frame,



Let P_1 and P_2 be of mass m, and the created particle be of mass M. We have

$$\boldsymbol{P}_1 + \boldsymbol{P}_2 = \boldsymbol{P}_1' + \boldsymbol{P}_2' + \boldsymbol{P}_M. \tag{499}$$

Since we are in the CoM frame,

$$\boldsymbol{p}_1 = -\boldsymbol{p}_2 \Rightarrow E_1 = E_2. \tag{500}$$

Squaring Eq · 499 gives

$$(\boldsymbol{P}_1 + \boldsymbol{P}_2)^2 = \frac{4E_1^2}{c^2} \quad \text{by } \boldsymbol{P}_1 = \begin{pmatrix} \frac{E_1}{c} \\ \boldsymbol{p}_1 \end{pmatrix}, \boldsymbol{P}_2 = \begin{pmatrix} \frac{E_2}{c} \\ \boldsymbol{p}_2 \end{pmatrix} \tag{501}$$

$$(\boldsymbol{P}_1' + \boldsymbol{P}_2' + \boldsymbol{P}_M)^2 = {\boldsymbol{P}_1'}^2 + {\boldsymbol{P}_2'}^2 + {\boldsymbol{P}_M}^2 + 2\boldsymbol{P}_1' \cdot \boldsymbol{P}_2' + 2\boldsymbol{P}_1' \cdot \boldsymbol{P}_M + 2\boldsymbol{P}_2' \cdot \boldsymbol{P}_M. \tag{502}$$

Note that

$${\boldsymbol{P}_1'}^2 + {\boldsymbol{P}_2'}^2 + {\boldsymbol{P}_M}^2 = 2m^2c^2 + M^2c^2. \tag{503}$$

Lemma 7.3

With variables as defined above,

$$\boldsymbol{P}_1 \cdot \boldsymbol{P}_2 \geqslant m_1 m_2 c^2. \tag{504}$$

Proof. $\boldsymbol{P}_1 \cdot \boldsymbol{P}_2$ is invariant, so we can work in any frame. In the rest frame of $\boldsymbol{P}_2$,

$$\boldsymbol{P}_1 = \begin{pmatrix} \frac{E_1}{c} \\ \boldsymbol{p}_1 \end{pmatrix}, \quad \boldsymbol{P}_2 = \begin{pmatrix} m_2 c \\ \boldsymbol{0} \end{pmatrix}. \tag{505}$$

Hence

$$\boldsymbol{P}_1 \cdot \boldsymbol{P}_2 = m_2 E_1. \tag{506}$$

Since $E_1 = \sqrt{p_1^2 c^2 + m_1^2 c^4} \geqslant m_1 c^2$, we get

$$\boldsymbol{P}_1 \cdot \boldsymbol{P}_2 \geqslant m_1 m_2 c^2. \quad \cdot 507$$

Note that we chose the + sign for the square root. The other sign leads to wrong non-relativistic limit, *i.e.* the kinetic energy would be negative. Recall that proper orthochronous Lorentz transformation will preserve positivity of E_1.

Now, applying the lemma to Eq · 502, we have

$$(\boldsymbol{P}_1' + \boldsymbol{P}_2' + \boldsymbol{P}_M)^2 \geqslant 2m^2c^2 + M^2c^2 + 2\left(m^2c^2 + 2mMc^2\right) = 4\left(m + \frac{1}{2}M\right)^2 c^2. \quad \cdot 508$$

Using Eq · 501, we have

$$\frac{4E_1^2}{c^2} \geqslant 4\left(m + \frac{1}{2}M\right)^2 c^2 \quad \cdot 509$$

$$E_1 \geqslant \left(m + \frac{1}{2}M\right)c^2 \quad \cdot 510$$

$$\underbrace{E_1 - mc^2}_{\substack{\approx \text{ kinetic energy} \\ \text{of each incoming} \\ \text{particle}}} \geqslant \frac{1}{2} \underbrace{Mc^2}_{\substack{\text{rest mass} \\ \text{energy of} \\ \text{created particle}}}. \quad \cdot 511$$

i.e. the total kinetic energy of the incoming particles

$$2\left(E_1 - mc^2\right) \quad \cdot 512$$

must be greater than the rest mass energy of the created particle, Mc^2.

7.16 Accelerated Motion in Special Relativity

We may define the acceleration 4-vector, for a massive particle,

$$\boldsymbol{A} := \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\tau}. \quad \cdot 513$$

Since $\boldsymbol{U} \cdot \boldsymbol{U} = c^2$, we have

$$\boldsymbol{U} \cdot \boldsymbol{A} = 0. \quad \cdot 514$$

Since $\boldsymbol{U} = \gamma\binom{c}{\boldsymbol{u}}$, using chain rule $\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t} = \gamma\frac{\mathrm{d}}{\mathrm{d}t}$, we have

$$\boldsymbol{A} = \gamma\begin{pmatrix} \dot{\gamma}c \\ \dot{\gamma}\boldsymbol{u} + \gamma\boldsymbol{a} \end{pmatrix} \quad \text{where} \quad \dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}, \quad \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}. \quad \cdot 515$$

We would like to consider motion for a constant acceleration, but we need to define a frame in which the acceleration is constant.

We will take an inertial frame S' that, at some moment in time, is instantaneously travelling at the same speed as the particle, *i.e.* $\boldsymbol{u}' = \boldsymbol{0}$. Hence $\gamma = 1$ and $\dot{\gamma} = 0$ in that frame.

[More precisely,

$$\dot{\gamma} = \frac{+\boldsymbol{u} \cdot \dot{\boldsymbol{u}}}{c^2\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} = 0 \quad \cdot 516$$

given that $u = 0$ in that frame.]

To make calculation easier, we will consider motion in one spatial dimension, so that in S' we have

$$\mathbf{A}' = \begin{pmatrix} 0 \\ a' \end{pmatrix}. \qquad \cdot 517$$

We say the acceleration is constant if a' is constant. [Later we will see that this follows from a constant force.]

We can get $\mathbf{A}$ from $\mathbf{A}'$ by using inverse Lorentz transformation Eq · 395 and Eq · 396, by speed u,

$$\mathbf{A} = \begin{pmatrix} \frac{\gamma u a'}{c} \\ \gamma a' \end{pmatrix}. \qquad \cdot 518$$

Matching with the general expression for $\mathbf{A}$,

$$\cancel{\gamma}\dot{\gamma} c = \frac{\cancel{\gamma} u a'}{c} \qquad \cdot 519$$

$$\cancel{\gamma}(\dot{\gamma} u + \gamma a) = \cancel{\gamma} a' \qquad \cdot 520$$

We can hence solve for $\dot{\gamma}$ and hence a, giving

$$a := \dot{u} = \left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}} a'. \qquad \cdot 521$$

Lecture 23 · 2026-03-17

Then

$$\int \frac{\mathrm{d}u}{\left(a'\left(1 - \frac{u^2}{c^2}\right)\right)^{\frac{3}{2}}} = \int \mathrm{d}t \qquad \cdot 522$$

$$u = \frac{a' c t}{\sqrt{c^2 + a'^2 t^2}}. \qquad \cdot 523$$

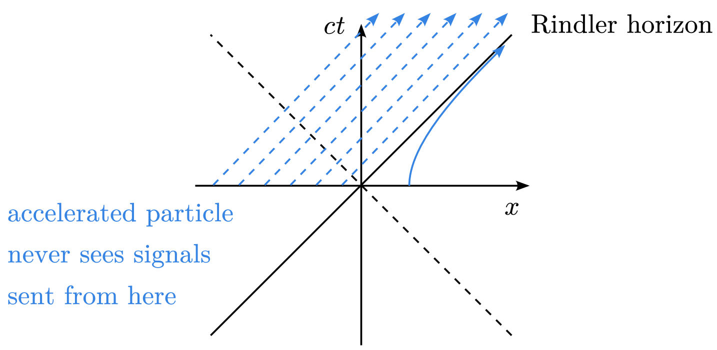
For early times $a't \ll c$, we get $u \approx a't$.

For late times $a't \gg c$, we get $u \to c$.

Since $u = \dot{x}$, we can integrate again to get

$$x = \frac{c}{a'}\sqrt{c^2 + a'^2 t^2} + \text{constant}. \qquad \cdot 524$$

This is the equation of an hyperbola. An accelerated particle follows a hyperbola in spacetime.



The Rindler horizon separates the accessible and inaccessible regoins of sapce time. [Black holes work in a similar way: one need to accelerate not to fall in.]

Let the force 4-vector $\boldsymbol{F}$ obey

$$\boldsymbol{F} := \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}\tau} = m\boldsymbol{A}. \qquad \cdot\ 525$$

Let us parameterise $\boldsymbol{F}$ as

$$\boldsymbol{F} := \begin{pmatrix} F^0 \\ \gamma \boldsymbol{f} \end{pmatrix}. \qquad \cdot\ 526$$

Recall that $\boldsymbol{P} = \begin{pmatrix} \frac{E}{c} \\ \boldsymbol{p} \end{pmatrix}$. For the spatial component of Eq · 525,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{1}{\gamma}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} = \boldsymbol{f}. \qquad \cdot\ 527$$

Hence $\boldsymbol{f}$ is the force appearing in Newton's law, but note that $\boldsymbol{p} = \gamma m \boldsymbol{u}$ is the relativistic 3-momentum. This connects to previous discussion that particles get heavier as they speed up.

Also, if $\boldsymbol{f}$ is constant [*i.e.* a constant force], then $\boldsymbol{a}'$ is constant in $\boldsymbol{A}$, *i.e.* indeed a constant $\boldsymbol{f}$ produces a constant acceleration in the instantaenous rest frame.

Consider the time component of Eq · 525,

$$F^0 = \frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}\tau} = \frac{\gamma}{c}\frac{\mathrm{d}E}{\mathrm{d}t}. \qquad \cdot\ 528$$

So F^0 is proportional to change of energy with time.

To recover a familiar equation, using Eq · 454,

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}(\boldsymbol{P}\cdot\boldsymbol{P}) = 2\boldsymbol{P}\cdot\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}\tau} \qquad \cdot\ 529$$

$$= 2\left(\frac{E}{c^2}\frac{\mathrm{d}E}{\mathrm{d}\tau} - \boldsymbol{p}\cdot\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}\right) \qquad \cdot\ 530$$

$$= 2\gamma^2 m\left(\frac{\mathrm{d}E}{\mathrm{d}t} - \boldsymbol{u}\cdot\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right) \qquad \cdot\ 531$$

Hence,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \boldsymbol{u}\cdot\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{u}\cdot\boldsymbol{f}. \qquad \cdot\ 532$$

which essentially says that the change in energy is the rate of work done.

7.17 Example Lorentz Force [Non-Examinable]

Lorentz force can be written in the form

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}\tau} = \frac{q}{c}\mathbf{G}\cdot\boldsymbol{U} \qquad \cdot\ 533$$

where $\mathbf{G}$ is the electromagnetic 4-tensor,

$$\mathbf{G} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}. \quad \cdot 534$$

The time component of Eq · 533 is

$$\gamma \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{q}{c}\boldsymbol{E} \cdot \gamma \boldsymbol{u} \quad \cdot 535$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{q}{c}\boldsymbol{E} \cdot \boldsymbol{u} \quad \cdot 536$$

i.e. the electric field does work, but magnetic field does not.

The spatial component of Eq · 533 is

$$\gamma \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \frac{q}{c}\left(\mathbf{G}^{i0}\boldsymbol{u}_0 - \mathbf{G}^{ij}\boldsymbol{u}_j\right) \quad \cdot 537$$

$$= \frac{q}{c}\left(\boldsymbol{E}^i \gamma c + \varepsilon^{ijk}\boldsymbol{B}^k c\gamma \boldsymbol{u}_j\right). \quad \cdot 538$$

Therefore

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{q}{c}(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}). \quad \cdot 539$$

i.e. the Lorentz force is relativistically invariant, but we need to account for the mixing of $\boldsymbol{E}$ and $\boldsymbol{B}$ under Lorentz transformations, where

$$\boldsymbol{u}' = \Lambda \boldsymbol{u} \quad \text{and} \quad \mathbf{G}' = \Lambda \mathbf{G} \Lambda^{-1}. \quad \cdot 540$$

END OF DOCUMENT