Part IA
Differential Equations

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These are Zixuan's notes for **Part IA – Differential Equations** at the University of Cambridge in 2025. The notes are not endorsed by the lecturers or the University, and all errors are my own.

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Syllabus and Overview

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Basic Calculus [5 Lectures]

Informal treatment of differentiation as a limit, the chain rule, Leibnitz's rule, Taylor series, informal treatment of O and o notation and l'Hôpital's rule; integration as an area, fundamental theorem of calculus, integration by substitution and parts. [3]

Informal treatment of partial derivatives, geometrical interpretation, statement (only) of symmetry of mixed partial derivatives, chain rule, implicit differentiation. Informal treatment of differentials, including exact differentials. Differentiation of an integral with respect to a parameter. [2]

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Equations with constant coefficients: exponential growth, comparison with discrete equations, series solution; modelling examples including radioactive decay.

Equations with non-constant coefficients: solution by integrating factor.

Nonlinear First-Order Equations [4 Lectures]

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Higher-Order Linear Differential Equations [8 Lectures]

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Multivariate Functions: Applications [5 Lectures]

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Simple examples of first- and second-order partial differential equations, solution of the wave equation in the form $f(x+ct)+g(x-ct)$.

Lecture 1 · 2025-10-10

Differential equations appear in almost all branches of science and applied mathematics.

For example, we have the following differential equation,

$$\underbrace{m}_{\text{mass of particle}} \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \underbrace{F}_{\text{force}}(x, t)$$

which relates the rate of change of position x, the *dependent variable*, with time t, the *independent variable*.

The main purpose of this course is to solve such equations.

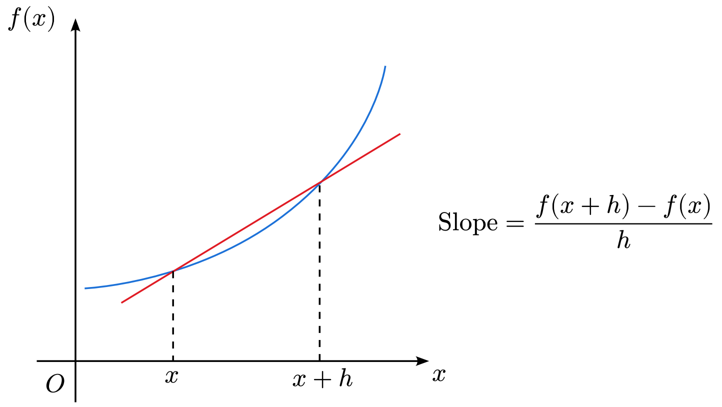
1 Differentiation

1.1 Introduction

Definition 1.1 (Derivative)

The **derivative** of a function $f(x)$ w.r.t. its argument x is the function

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$



Remark. For the derivative to exist, we require the left-handed and right-handed limit to exist and be equal. We write

$$\lim_{h \to 0^-} \text{ and } \lim_{h \to 0^+}$$

for the limits. *e.g.* $|x|$ is **not** differentiable at $x = 0$, despite being differentiable at everywhere else.

Informally, if $\lim_{x \to x_0} f(x) = A$, then $f(x)$ can be made arbitrarily close to A by making x sufficiently close to x_0. Nonetheless, we do not require $f(x_0) = A$. [See IA Analysis I for a more formal discussion.]

Notation. We write

$$\frac{df}{dx} = f'(x) = \underbrace{\dot{f}(x)}_{\text{usually used for } f(t)}.$$

We can higher derivatives for sufficiently smooth functions. For example,

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2} = f''(x) = \ddot{f}(x).$$

To refer to the nth derivative, we write

$$\frac{d^n f}{dx^n} = f^{(n)}(x).$$

1.2 Big O and Little O

If we wish to compare the behavior of functions close to a limiting point x_0, we can use **order parameters**. There are two kinds of them: Big O and Little O.

1. **Big O** — "can be bounded by"

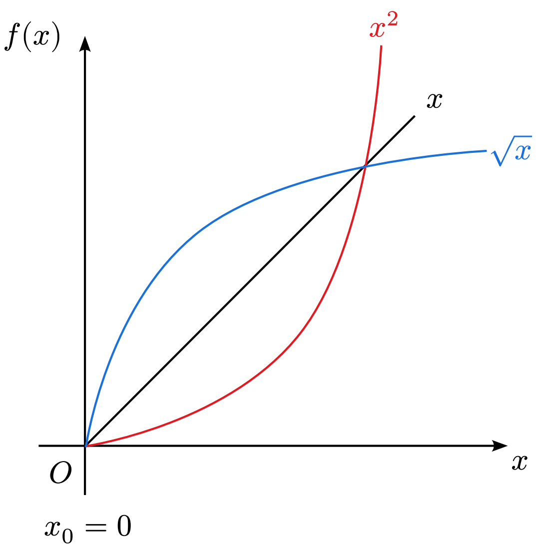
Definition 1.2 (Big O)

- if x_0 is finite, then $f(x)$ is $O(g(x))$ as $x \to x_0$ if $\exists \delta > 0$ and $M > 0$ such that $\forall x$ with $0 < |x - x_0| < \delta$, we have

$$|f(x)| \leqslant M|g(x)|.$$

  We often write $f(x) \underbrace{=}_{\text{abuse of notation}} O(g(x))$. It follows that $f(x)g(x))$ is bounded as $x \to x_0$.

  e.g. if $x_0 = 0$, then $x \neq O\left(x^2\right), x^2 = O(x)$ and $x = O\left(\sqrt{x}\right)$.



e.g. $\sin 2x = O(x)$ as $x \to 0$ since $|\sin 2x| \leqslant 2|x|$.

- if $x_0 = \infty$, then $f(x)$ is $O(g(x))$ as $x \to \infty$ if $\exists x_1 \in \mathbb{R}$ and $M > 0$ such that $\forall x > x_1$, $|f(x)| \leqslant M|g(x)|$.

  e.g. $2x^3 + 2x = O\left(x^3\right)$ as $x \to \infty$ since $\forall x > 1$, $\left|2x^3 + 4x\right| \leqslant 2\left|x^3\right| + 4|x| \leqslant 6\left|x^3\right|$

1. **Little O** — "much smaller than"

Definition 1.3 (Little O)

$f(x)$ is $o(g(x))$ as $x \to x_0$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall |x - x_0| < \delta$,

$$|f(x)| \leqslant \varepsilon |g(x)|.$$

If $g \neq 0$ in vicinity of x_0 (regardless of the behavoir at 0), equivalently

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

We often write $f(x) = o(g(x))$.

e.g. $x^2 = o(x)$ as $x \to 0$, since $\lim_{x \to 0}\left(\frac{x^2}{x}\right) = 0$.

e.g. $\sqrt{x} = o(x)$ as $x \to \infty$.

Remark.

- $f(x) = o(g(x))$ is a stronger statement than $f(x) = O(g(x))$. Big O shows that a function is bounded by a **given** multiple, whereas Little O shows that it is bounded by **any** multiple.

  So, $f(x) = o(g(x)) \Rightarrow f(x) = O(g(x))$, but not the converse.

  e.g. $2x = O(x)$ but $2x \neq o(x)$ as $x \to 0$.
- Multiplicative constants do not matter for Big O. *i.e.* If $f(x) = O(g(x))$, $af(x) = O(g(x))$ and $f(x) = O(ag(x))$ for any non-zero constant a.

Lecture 2 · 2025-10-13

Order parameters are useful to classify the remainder terms before taking limits. So we can write

$$f(x_0 + h) - f(x_0) = hf'(x_0) + \underbrace{\varepsilon(h)}_{\text{remainder}}$$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) + \lim_{h \to (0)} \frac{\varepsilon(h)}{h}.$$

Therefore $\varepsilon(g) = o(h)$ as $h \to 0$.

Hence

$$f(x_0 + h) - f(x_0) = hf'(x_0) + o(h)$$

as $h \to 0$. This result extends to Theorem 1.10 (Taylor's Theorem).

1.3 Rules for Differentiation

Theorem 1.4 (Chain rule)

[Differentiating a function of a function.] Given $f(x) = F(g(x))$, then

$$\frac{df}{dx} = F'(g(x))\frac{dg}{dx} = \frac{dF}{dg}\frac{dg}{dx}.$$

Theorem 1.5 (Product rule)

Given $f(x) = u(x)v(x)$, then

$$\frac{df}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Theorem 1.6 (Quotient rule)

[Special case of the product rule.] Given $f(x) = \frac{u(x)}{v(x)}$, then

$$\frac{df}{dx} = \frac{vu' - uv'}{v^2}.$$

Consider $f(x) = u(x)v(x)$. By repeated applying the product rule, we have

$$\begin{aligned} f' &= u'v + v'u \\ f'' &= u''v + \underbrace{u'v' + v'u'}_{2u'v'} + v''u \\ f''' &= u'''v + \underbrace{u''v' + 2u''v'}_{3u''v'} + \underbrace{'v'' + v''u'}_{3u'v''} + v'''u. \end{aligned}$$

c.f. Pascal's triangle, we can generalize this into Leibniz's Rule.

Theorem 1.7 (Leibniz's rule)

[Generalization of the product rule.] Given $f(x) = u(x)v(x)$, we have

$$f^{(n)}(x) = \sum_{n=0}^{n} \binom{n}{r} u^{(n-r)}(x)v^{(r)}(x).$$

1.4 Taylor Series

Definition 1.8 (Taylor series)

For a function $f(x)$ which is infinitely differentiable at $x = x_0$, the Taylor series about x_0 is

$$T_f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2!}(x - x_0)^2 f''(x_0) + \frac{1}{3!}(x - x_0)^3 f'''(x_0) + \ldots$$

Definition 1.9 (Taylor polynomial)

For a function $f(x)$ which is infinitely differentiable at $x = x_0$, the **Taylor polynomial** of degree n is

$$P_n(x) = f(x_0) + (x - x_0)f'(x_0) + \ldots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0).$$

Note that $P_n(x)$ are the partial sums of $T_f(x)$.

Theorem 1.10 (Taylor's theorem)

For a function $f(x)$ which is differentiable n times at $x = x_0$,

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{1}{2!}h^2 f''(x_0) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + \underbrace{E_n}_{\text{remainder}}$$

where $E_n = o(h^n)$ as $h \to 0$.

Corollary 1.11 (Stronger version of Taylor's theorem)

Following Theorem 1.10, if $f^{(n+1)}(x)$ exists $\forall x \in (x_0, x_0 + h)$ and $f^{(n+1)}(x)$ is continuous in this range, then

$$\begin{aligned} E_n &= O\left(h^{n+1}\right) \quad \text{as } n \to 0 \\ &= f^{n+1}(x_n)\frac{h^{n+1}}{(n+1)!}. \end{aligned}$$

for some x_n with $x_0 \leqslant x_n \leqslant x_0 + h$.

Remark. Note that $E_n = O\left(h^{n+1}\right)$ is a stronger statement than $o(h^n)$.

e.g. $h^{n+\frac{1}{2}}$ is $o(h^n)$ but not $O\left(h^{n+1}\right)$ as $h \to 0$.

With $x = x_0 + h$, Theorem 1.10 (Taylor's theorem) gives

$$f(x) = P_n(x) + E_n.$$

This is to say, that $P_n(x)$ provides a local approximation to $f(x)$ in the vicinity of x_0 with error $o(h^n)$ or $O\left(h^{n+1}\right)$.

Corollary 1.12

If $\lim_{n\to\infty} E_n = 0$, then the Taylor series converges to $f(x)$.

Example 1.13

Consider, about $x_0 = 0$, the function $f(x) = \exp(x)$. Then for $h > 0$,

$$E_n = \frac{h^n + 1}{(n+1)!}\exp(x_n)$$

where $0 \leqslant x_n \leqslant h$.

Then the fractional error

$$\begin{aligned}\frac{E_n}{\exp(h)} &= \frac{h^n+1}{(n+1)!}\underbrace{\exp(x_n-h)}_{\leqslant 1 \text{ and } >0} \\ &\leqslant \frac{h^{n+1}}{(n+1)!}.\end{aligned}$$

Therefore, for a given target accurarcy at $x = h$, this can be used to specify how large n must be.

1.5 L'Hôpital's Rule

L'Hôpital's rule allows us to deal with limits of indeterminate forms.

Theorem 1.14 (L'Hôpital's rule)

Let $f(x)$ and $g(x)$ be differentiable at x_0 with continuous first derivatives there, and

$$\begin{aligned}\lim_{x\to x_0} f(x) &= f(x_0) = 0 \\ \lim_{x\to x_0} g(x) &= g(x_0) = 0,\end{aligned}$$

then if $g'(x_0) \neq 0$,

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

provided that the limit on the RHS exists.

Proof. From Theorem 1.10 (Taylor's theorem) we have

$$\begin{aligned}f(x) &= f(x_0) + (x-x_0)f'(x) + o(x-x_0) \\ g(x) &= g(x_0) + (x-x_0)g'(x) + o(x-x_0)\end{aligned}$$

as $x \to x_0$.

Thus

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x_0)+\frac{o(x-x_0)}{x-x_0}}{\underbrace{g'(x_0)}_{\neq 0}+\frac{o(x-x_0)}{x-x_0}}$$
$$= \frac{\lim_{x\to x_0}\left(f'(x_0)+\frac{o(x-x_0)}{x-x_0}\right)}{\lim_{x\to x_0}\left(g'(x_0)+\frac{o(x-x_0)}{x-x_0}\right)}$$
$$= \frac{f'(x_0)}{g'(x_0)} \quad \text{by definition of the little O}$$
$$= \lim_{x\to x_0}\frac{f'(x)}{g'(x)} \quad \text{from the continuity of 1st derivatives.}$$

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Remark. L'Hôpital's rule can be generalized. For example, if $f'(x_0) = g'(x_0) = 0$, then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f''(x)}{g''(x)}.$$

This can be generalized even further to produce a limit on RHS results.

Example 1.15

Let

$$f(x) = 3\sin x - \sin 3x,$$
$$g(x) = 2x - \sin 2x.$$

Then

$$f'''(x) = -3\cos x + 27\cos 3x,$$
$$g'''(x) = 8\cos 2x.$$

Therefore, we can find the limit

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'''(x)}{g'''(x)}$$
$$= \frac{24}{3}$$
$$= 8.$$

2 Integration

We have seen the definition of integration in terms of *area under a curve* and *inverse of differentiation*. We shall review what these really mean.

2.1 Integrals as Riemann Sums

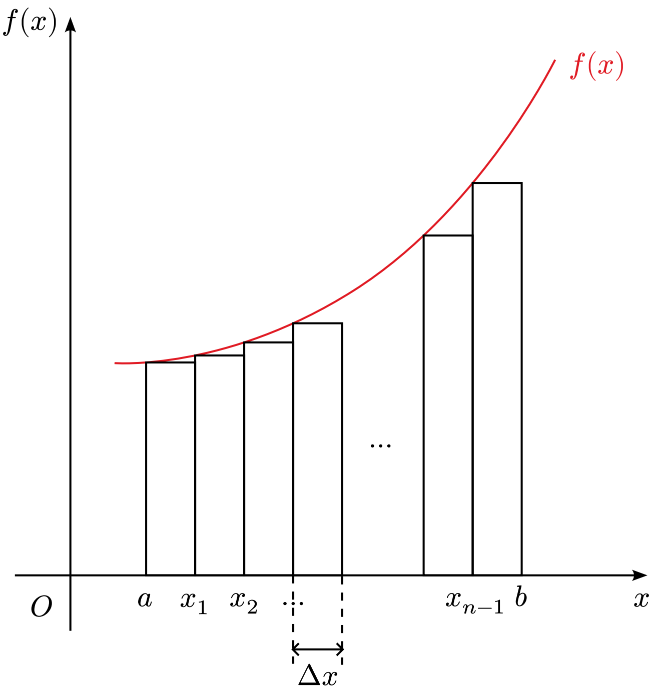
This section talks about the *area under a curve* idea.

Definition 2.1 (Integral as Riemann sum)

The **integral** of a suitably well-defined function $f(x)$ is the limit of a sum

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \sum_{n=0}^{N-1} f(x_n)\Delta x$$

where $\Delta x = \frac{b-a}{N}$ and $x_n = a + n\Delta x$.

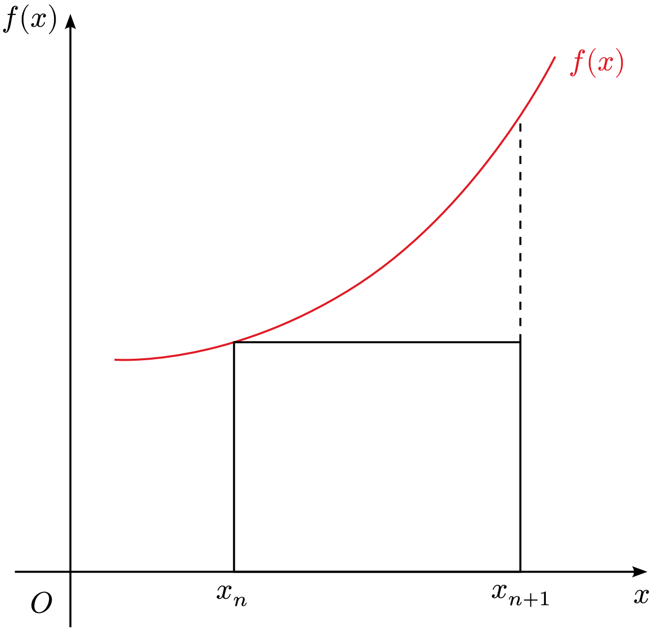


Definition 2.2 (Riemann integrable)

Following Definition 2.1, a function $f(x)$ is **Riemann integrable** if the generalized Riemann sum does not depend on exactly how we choose the rectangles in the limit that all $\Delta x \to 0$.

[This includes the cases where Δx is non-uniform, or we cannot evaluate f on the LHS, *etc*.]

How are we sure this limit evaluates to the *area under a curve* idea? We shall consider one rectangle at first.



We will need to borrow this following idea from IA Analysis I for now:

Theorem 2.3 (Mean-value theorem (MRT) [IA Analysis I content])

For a continuous function $f(x)$, the area under curve from x_n to x_{n+1} is

$$A_n(x_{n+1} - x_n)f(c_0)$$

for some c_n where $x_n \leqslant c_n \leqslant x_n + 1$.

Hence, if $f(x)$ is differentiable, then by Taylor's theorem

$$\begin{aligned} f(c_n) &= f(x_n) + O(c_n - x_n) \quad \text{as } c_n - x_n \to 0 \\ &= f(x_n) + O(\Delta x) \quad \text{since } \Delta x \geqslant c_n - x_n. \end{aligned}$$

Hence,

$$A_n = \Delta x f(x_0) + O\left((\Delta x)^2\right).$$

Therefore, the total area under curve from $x = a$ to $x = b$ is

$$\begin{aligned} A &= \lim_{N \to \infty} \sum_{n=0}^{N-1} A_n \\ &= \underbrace{\lim_{N \to \infty} \sum_{n=0}^{N-1} f(x_n)\, \Delta x}_{\text{definition of integral}} + \underbrace{\lim_{N \to \infty} N \cdot O\left(\left(\frac{b-a}{N}\right)^2\right)}_{O(\frac{1}{N})} \\ &= \int_a^b f(x)\, \mathrm{d}x. \end{aligned}$$

2.2 Fundamental Theorem of Calculus (FTC)

This section talks about the *inverse of differentiation* idea.

Theorem 2.4 (Fundamental Theorem of Calculus)

Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$ for some Riemann integrable function $f(t)$. Then

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(t)\,\mathrm{d}t\right] = f(x),$$

Proof. By definition,

$$\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}x} &= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int_x^{x+h} f(t)\,\mathrm{d}t\right] \quad \text{by considering the integral as a sum} \\
&= \lim_{h\to 0}\frac{1}{h}\left[f(x)h + O\left(h^2\right)\right] \quad \text{from MVT and Taylor's theorem} \\
&= \lim_{h\to 0}[f(x) + O(h)] \\
&= f(x).
\end{aligned}$$

Remark. $F(x)$ is a solution to the differential equation

$$\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$$

with $F(a) = 0$.

Corollary 2.5

Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$ for some Riemann integrable function $f(t)$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b f(t)\,\mathrm{d}t = -f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) = \frac{\mathrm{d}F}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} = f(g(x))\frac{\mathrm{d}g}{\mathrm{d}x}.$$

Notation. Indefinite integrals are written in the form

$$\int f(x)\,\mathrm{d}x \quad \text{or} \quad \int^x f(t)\,\mathrm{d}t.$$

The unspecified lower limit gives an integration constant.

2.3 Integration Techniques

2.3.1 Integration by Substitution

If the integrand contains a function of a function, it *might* help to substitute for the inner function.

Example 2.6

Consider $I = \int \frac{1-2x}{\sqrt{x-x^2}}\,dx$. Then let

$$u = x - x^2 \quad \Rightarrow \quad \frac{du}{dx} = 1 - 2x.$$

and

$$I = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{x - x^2} + C.$$

Another important class of substitutions are trigonometric substitutions. These make use of the following identities:

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 \\
1 + \tan^2\theta &= \sec^2\theta \\
\cosh^2 u - \sinh u &= 1 \\
1 - \tanh^2 u &= \operatorname{sech}^2 u.
\end{aligned}$$

We can use the following table as a substitution reference.

Integrand contains	Substitution
$\sqrt{1-x^2}$	$x = \sin\theta$
$1+x^2$	$x = \tan\theta$
$\sqrt{x^2+1}$	$x = \sinh u$
$\sqrt{x^2-1}$	$x = \cosh u$
$1-x^2$	$x = \tanh u$

Example 2.7

Consider

$$\begin{aligned}
I &= \int \sqrt{2x - x^2}\,dx \\
&= \int \sqrt{1 - (x-1)^2}.
\end{aligned}$$

We should try $x - 1 = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$ with $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$, so θ would be unique for $0 \leqslant x \leqslant 2$.

$$\begin{aligned} I &= \int \sqrt{\cos^2\theta}\cos\theta \, d\theta \\ &= \int \cos^2\theta \, d\theta \\ &= \dots \\ &= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + C. \end{aligned}$$

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2.3.2 Integration by Parts

Recall the product rule

$$(uv)' = u''v + uv',$$

we can derive the integration by parts technique

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Example 2.8

Let

$$I = \int_0^\infty \underbrace{x}_{u}\, \underbrace{e^{-x}}_{v'} \, dx.$$

Then we have

$$\begin{aligned} I &= [-xe^{-x}]_0^\infty - \int_0^\infty (-e^{-x}) \, dx \\ &= [-e^{-x}]_0^\infty \\ &= 1. \end{aligned}$$

Example 2.9

Let $I = \int \ln x \, dx$. We can let $u = \ln x$ and $v' = 1$, and then

$$\begin{aligned} I &= x\ln x - \int \frac{x}{x} \, dx \\ &= x\ln x - \int 1 \, dx \\ &= x\ln x - x + C. \end{aligned}$$

This integration by parts method also works for inverse trignometric functions and inverse hyperbolic functions.

3 Partial Differentiation

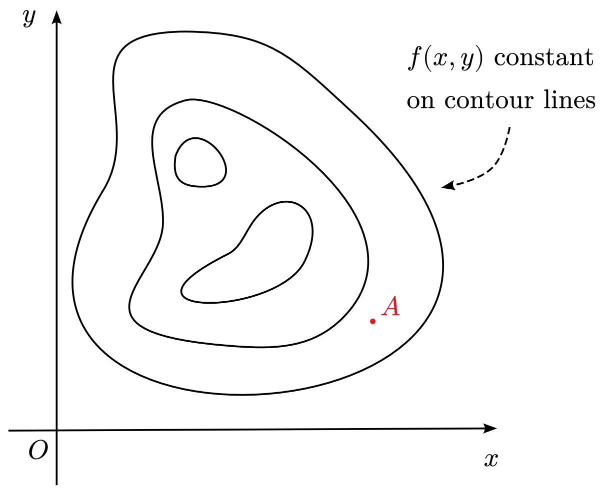
3.1 Functions of Several Variables

We shall now generalize to functions of more than one independent variables, which are also called **multivariate functions**.

Example 3.1 (Examples of multivarate functions)

- Height of terrain, $h(x,y)$
- Temperature in a room, $T(x,y,z,t)$
- Pressure of gas as a function of volume and temperature, $p(V,T)$

A way of representing these functions is using a **contour plot**.



The slope of a point A on the plot depends on the direction. Firstly, let us consider what happens along the coodinate directions.

3.2 Partial Derivatives

Partial derivatives ar ederivatives of multivariate functions with respect to one variable, while holding the other fixed.

Definition 3.2 (Partial derivative)

Given a function of several variables, *e.g.* $f(x,y)$, the **partial derivative** of f with respect to x at fixed y is

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}.$$

Roughly speaking, this is the slope of f when moving in the positive x direction.

$$\left.\frac{\partial f}{\partial y}\right|_x$$

can be defined similarly.

Example 3.3

Consider $f(x, y) = x^2 + y^3 + e^{xy^2}$. Then

$$\left.\frac{\partial f}{\partial x}\right|_y = 2x + 0 + y^2 e^{xy^2},$$

$$\left.\frac{\partial f}{\partial y}\right|_x = 0 + 3y^2 + 2xye^{xy^2}.$$

We can similarly calculate higher derivatives.

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_y = 2 + y^4 e^{xy^2}.$$

Also, we can do mixed derivatives.

$$\left.\frac{\partial}{\partial x}\left(\left.\frac{\partial f}{\partial y}\right|_x\right)\right|_y = 2ye^{xy^2} + 2xy^3 e^{xy^2},$$

$$\left.\frac{\partial}{\partial y}\left(\left.\frac{\partial f}{\partial x}\right|_y\right)\right|_x = 2ye^{xy^2} + 2xy^3 e^{xy^2}.$$

Notation. The notation is getting somewhat cumbersome. We usually omit $|_x$ and use of ∂ to indicate that *all other* variables are kept fixed. *i.e.*

$$\left.\frac{\partial}{\partial x}\left(\left.\frac{\partial f}{\partial y}\right|_x\right)\right|_y \equiv \frac{\partial^2 f}{\partial x \partial y}.$$

Alternatively, there is a even more compact notation:

$$f_x \equiv \frac{\partial f}{\partial x}, \quad f_{xy} \equiv \frac{\partial^2 f}{\partial y \partial x}.$$

Note that in Example 3.3, it seems that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

This is indeed the case under certain conditions.

If f is a multivariate function with continuous 2nd derivatives, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

if the conditions of Schwarz's theorem is satisfied.

3.3 Multivariate Chain Rule

Give path $x(t), y(t)$ and a function $f(x,y)$, consider $\frac{df}{dt}$ along the path.

Theorem 3.4 (Differential form of chain rule for partial derivatives)

Differential df of a function $f(x,y)$ is

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy.$$

Proof. Consider the change in f under

$$(x,y) \to (x+\delta x, y+\delta y).$$

Then

$$\begin{aligned}\delta f &= f(x+\delta x, y+\delta y) - f(x,y) \\ &= [f(x+\delta x, y+\delta y) - f(x+\delta x, y)] + [f(x+\delta x, y) - f(x,y)].\end{aligned}$$

Conside the two brackets separately. Then, as $\delta x, \delta y \to 0$,

$$\begin{aligned} f(x+\delta x, y) - f(x,y) &= f_x(x,y)\delta x + o(\delta x), \quad \text{by Taylor's theorem} \\ f(x+\delta x, y+\delta y) - f(x+\delta y, y) &= f_y(x+\delta x, y)\delta y + o(\delta y).\end{aligned}$$

Note that

$$f_y(x+\delta x, y) = f_{y(x,y)} + f_{yx}(x,y)\delta x + o(\delta x).$$

Hence,

$$\delta f = \left[f_y(x,y) + f_{yx}(x,y)\delta x + o(\delta x)\right]\delta y + f_x(x,y)\delta x + o(\delta y) + o\delta(x).$$

Taking the limit $\lim_{\delta x, \delta y \to 0}$ gives the required results.

Thus, for the path $x(t), y(t)$, we have

$$\begin{aligned}\frac{d}{dt} f(x(t), y(t)) &= \lim_{\delta x, \delta y, \delta t \to 0}\left[\frac{\partial f}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\frac{\delta y}{\delta t}\right] \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.\end{aligned}$$

If instead we parameterize path by coordinate x, so we are left with $y(x)$, then

$$\frac{d}{dx} f(x, y(x)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

We can also reach the integral form of chain rule:

Theorem 3.5 (Integral form of chain rule for partial derivatives)

For a function $f(x,y)$, the change in f between two endpoints is

$$\Delta f = \int \mathrm{d}f = \int \frac{\partial f}{\partial x}\,\mathrm{d}x + \int \frac{\partial f}{\partial y}\,\mathrm{d}y.$$

For $f(x(t), y(t))$,

$$\Delta f = \int \underbrace{\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}}_{\frac{\mathrm{d}f}{\mathrm{d}t}}\,\mathrm{d}t,$$

so the final result does not depend on a particular path for given endpoints.

3.4 Applications of Multivariate Chain Rule

3.4.1 Change of Variables

It is often useful to write a differential equation in a different coordinate system.

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For example, we can change Cartesian (x, y) coordinates to polar (r, θ) coordinates with

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Think of $f(x, y)$ as $f(x(r, \theta), y(r, \theta))$. From the chain rule,

$$\left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial r}\right|_\theta$$
$$\left.\frac{\partial f}{\partial \theta}\right|_r = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial \theta}\right|_r + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial \theta}\right|_r.$$

3.4.2 Implicit Differentiation

For $f(x, y, z)$, we have

$$\mathrm{d}f = \left.\frac{\partial f}{\partial x}\right|_{y,z}\mathrm{d}x + \left.\frac{\partial f}{\partial y}\right|_{x,z}\mathrm{d}y + \left.\frac{\partial f}{\partial z}\right|_{x,y}\mathrm{d}z.$$

Consider $f(x, y, z) = \text{constant}$, which represents a surface in 3D space. In this case, it implicitly defines

$$z = z(x, y), \quad x = (y, z), \quad y = (x, z),$$

but we may not be able to find the solutions explicitly.

However, we can still evaluate derivatives like $\left.\frac{\partial z}{\partial x}\right|_y$.

Example 3.6

In the equation

$$xy + y^2 z + z^5 = 1,$$

we cannot find $z(x,y)$ explicitly. However, we can take the derivative with respect to x holding y fixed only, we get

$$\text{LHS} = y + y^2 \left.\frac{\partial z}{\partial x}\right|_y + 5z^4 \left.\frac{\partial z}{\partial x}\right|_y = 0$$
$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{y}{y^2+5z^4}.$$

Remark. Be aware that $\left.\frac{\partial\,\text{LHS}}{\partial x}\right|_y \neq \left.\frac{\partial\,\text{LHS}}{\partial x}\right|_z$.

Theorem 3.7 (Cyclical rule)

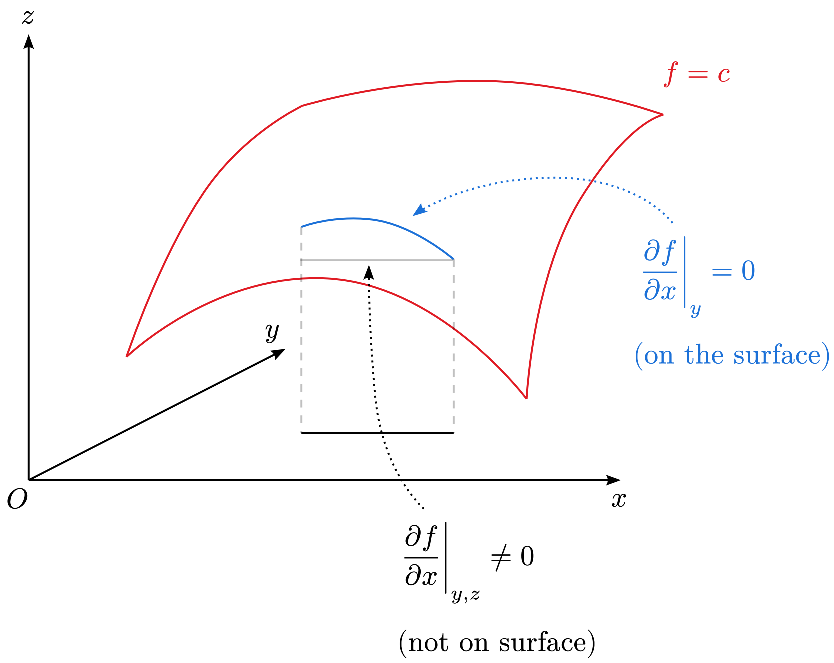
If $f(x,y,z) = \text{constant}$, then

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1.$$

Proof. In general, if $f(x,y,z) = \text{constant}$, then

$$0 = \mathrm{d}f = \left.\frac{\partial f}{\partial x}\right|_{y,z} \mathrm{d}x + \left.\frac{\partial f}{\partial y}\right|_{x,z} \mathrm{d}y + \left.\frac{\partial f}{\partial z}\right|_{x,y} \mathrm{d}z.$$

Therefore, we can't vary x, y, z independently and stay on the surface.



Thus (be aware that z is not fixed),

$$
\begin{aligned}
0 = \left.\frac{\partial f}{\partial x}\right|_y &= \left.\frac{\partial f}{\partial x}\right|_{y,z} \left.\frac{\partial x}{\partial x}\right|_y + \left.\frac{\partial f}{\partial y}\right|_{x,z} \left.\frac{\partial y}{\partial x}\right|_y + \left.\frac{\partial f}{\partial z}\right|_{x,y} \left.\frac{\partial z}{\partial x}\right|_y \\
0 = \left.\frac{\partial f}{\partial x}\right|_y &= \left.\frac{\partial f}{\partial x}\right|_{y,z} \cdot 1 + \left.\frac{\partial f}{\partial y}\right|_{x,z} \cdot 0 + \left.\frac{\partial f}{\partial z}\right|_{x,y} \left.\frac{\partial z}{\partial x}\right|_y \\
0 = \left.\frac{\partial f}{\partial x}\right|_y &= \left.\frac{\partial f}{\partial x}\right|_{y,z} + \left.\frac{\partial f}{\partial z}\right|_{x,y} \left.\frac{\partial z}{\partial x}\right|_y \\
\left.\frac{\partial z}{\partial x}\right|_y &= -\frac{\left.\frac{\partial f}{\partial x}\right|_{y,z}}{\left.\frac{\partial f}{\partial z}\right|_{x,y}}.
\end{aligned}
$$

We can similarly find $\left.\frac{\partial x}{\partial y}\right|_z$ and $\left.\frac{\partial y}{\partial z}\right|_x$.

Therefore, we can conclude

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1.$$

The **reciprocal rule** applies if the same variables are held fixed:

$$\left.\frac{\partial x}{\partial z}\right|_y = -\frac{\left.\frac{\partial f}{\partial z}\right|_{x,y}}{\left.\frac{\partial f}{\partial x}\right|_{y,z}}.$$

Therefore,

$$\left.\frac{\partial x}{\partial z}\right|_y = \frac{1}{\left.\frac{\partial z}{\partial x}\right|_y}.$$

Important. For the change in variables $(r, \theta) \to (x, y)$, we have

$$\left.\frac{\partial r}{\partial x}\right|_y = \frac{1}{\left.\frac{\partial x}{\partial r}\right|_y}$$

but

$$\left.\frac{\partial r}{\partial x}\right|_y \ne \frac{1}{\left.\frac{\partial x}{\partial r}\right|_\theta}.$$

3.5 Differentiation of an Integral w.r.t. a Parameter

Theorem 3.8

For a family of functions $f(x; c)$ where c is a parameter. Define

$$I(c) = \int_{a(c)}^{b(c)} f(x; c)\, dx.$$

Then

$$\frac{dI}{dc} = \int_{a(c)}^{b(c)} \frac{\partial}{\partial c} f(x;c)\,dx + f(b(c);c)\frac{db}{dc} - f(a(c);c)\frac{da}{dc}.$$

Proof. By definition,

$$\frac{dI}{dc} = \lim_{\delta c \to 0} \frac{1}{\delta c}\left[\int_{a(c+\delta c)}^{b(c+\delta c)} f(x;c+\delta c)\,dx - \underbrace{\int_{a(c)}^{b(c)} f(x;c)\,dx}_{(4)} \right].$$

Now let us consider the parts of the expression separately.

$$\int_{a(c+\delta c)}^{b(c+\delta c)} f(x;c+\delta c)\,dx = \underbrace{\int_{a(c)}^{b(c)}}_{(1)} + \underbrace{\int_{b(c)}^{b(c+\delta c)}}_{(2)} - \underbrace{\int_{a(c)}^{a(c+\delta c)}}_{(3)}$$

Hence

$$\begin{aligned}
(1)+(4) : &\lim_{\delta c \to 0} \frac{1}{\delta c}\left[\int_{a(c)}^{b(c)} (f(x;c+\delta c) - f(x;c))\,dx\right] \\
&= \int_{a(c)}^{b(c)} \frac{\partial f}{\partial c}(x;c)\,dx. \\
(2) : &\lim_{\delta c \to 0} \frac{1}{\delta c}\left[\underbrace{\int_{b(c)}^{b(c+\delta c)} f(x;c+\delta c)\,dx}_{[b(c+\delta c)-b(c)]f(\bar{x};c+\delta c) \text{ by MVT}}\right] \\
&= \frac{db}{dc} f(b(c);c) \\
(3) : &\text{ follows the same trick, and the result can be claimed.}
\end{aligned}$$

Example 3.9

Consider

$$I(\lambda) = \int_0^{\lambda} e^{-\lambda x^2}\,dx.$$

Then

$$\frac{dI}{d\lambda} = \int_0^{\lambda} -x^2 e^{-\lambda x^2}\,dx + e^{-\lambda^2}\frac{d\lambda}{d\lambda}.$$

Example 3.10

Suppose that we want to evaluate

$$J_n = \int_0^\infty x^n e^{-x} \, dx.$$

Then, let

$$I(\lambda) = \int_0^\infty e^{-\lambda x} \, dx = \frac{1}{\lambda}$$

with $\lambda > 0$. Hence

$$\frac{d^n I}{d\lambda^n} = \int_0^\infty (-x)^n e^{-\lambda x} \, dx = \frac{(-1)^n n!}{\lambda^{n+1}}.$$

> ***Important.*** Note that in this step we are still using Theorem 3.8, it is just that the limits are now 0 and ∞, which are independent of λ, so the last two terms vanish.

Setting $\lambda = 1$, we get $J_n = n!$.

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4 First-Order Linear Differential Equations

Definition 4.1 (nth Order Differential Equation)

A differential equation is of ***n*th order** if the highest derivative in the equation is of order n.

Definition 4.2 (Linear Differential Equation)

A differential equation is **linear** if the dependent variable y appears linearly.

Definition 4.3 (Ordinary Differential Equation)

A differential equation is **ordinary** if it contains one independent variable and its derivatives.

Example 4.4

$$x^2 y + y' = 0$$

is a first-order linear ordinary differential equation.

4.1 The Exponential Function

In order to introduce first-order linear ordinary differential equations, we first need to explore the exponential function.

Definition 4.5 (Exponential Function)

The **exponential function** is defined the infinite series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This can also be written as

$$\begin{aligned}\exp(x) &= \lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k \\ &= \lim_{k\to\infty} \left[1 + k\frac{x}{k} + \frac{k(k-1)}{2!}\left(\frac{x}{k}\right)^2 + \ldots\right] \quad \text{using the binomial theorem.}\end{aligned}$$

Differentiaing the exponential function, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) &= 1 + \frac{2}{2!}x + \frac{3}{3!}x^2 + \ldots \\ &= \exp(x).\end{aligned}$$

This allows us to define the exponential function in another way. We can define $\exp(x)$ to be the solution of

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f$$

with the initial condition $f(0) = 1$.

Proposition 4.6

The exponential function satisfies this key property:

$$\exp(a + b) = \exp(a)\exp(b).$$

Following this property, it suggests that we should write $\exp(x)$ as e^x, where

$$\mathrm{e} = \exp(1) = \lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k.$$

The inverse function $\ln x$ is defined as the function where

$$\exp(\ln x) = x.$$

It follows that $a^x = \left(\mathrm{e}^{\ln a}\right)^x = \mathrm{e}^{x\ln a}$

Hence

$$\frac{\mathrm{d}a^x}{\mathrm{d}x} = \frac{\mathrm{d}e^{x\ln a}}{\mathrm{d}x} = e^{x\ln a}\ln a = a^x\ln a.$$

Definition 4.7 (Eigenfunction)

An **eigenfunction** of an operator is a function that is unchanged up to multiplicative scaling by the **eigenvalue**, under action of the operator.

In case of the differential operator, an eigenfunction satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \lambda f(x).$$

Hence, the eigenfunctions of the differential operator are of the form

$$f(x) = C\mathrm{e}^{\lambda x}$$

4.2 First Order Linear Ordinary Differential Equations

Definition 4.8 (Homogeneous Differential Equation)

A differential equation is **homogeneous** if all terms involve the dependent variable or its derivatives. This implies that $y = 0$ is a solution (trivial solution).

Proposition 4.9

Here are some properties of first-order linear ordinary differential equations.

- Any nth order linear ODE has n independent solutions.
- For any linear homogenous ODE, any constant multiple of a solution is also a solution.

4.2.1 Homogeneous Linear ODEs with Constant Coefficients

Definition 4.10 (Constant Coefficients)

A differential equation has **constant coefficients** if the independent variable does not appear explicitly.

Proposition 4.11

Solutions of linear homogeneous ODEs with constant coefficients (for any order) are of the form $e^{\lambda x}$

Example 4.12

Consider the equation

$$5\frac{dy}{dx} - 3y = 0.$$

We should try $y = Ae^{\lambda x}$. Then

$$5\lambda Ae^{\lambda x} - 3Ae^{\lambda x} = 0.$$

Which leads to (given that $A \neq 0$)

$$5\lambda - 3 = 0 \quad \text{(characteristic equation)}$$

So $\lambda = \frac{3}{5} \Rightarrow y = Ae^{\frac{3x}{5}}$.

This is the general solution, as it contains an arbitrary constant A.

To specify unique solutions, it requires us to apply suitable initial conditions. [n conditions are needed for a nth order ODE.]

For example, in Example 4.12, if we have $y(0) = y_0$, then $A = y_0$.

4.2.2 Discrete Equations

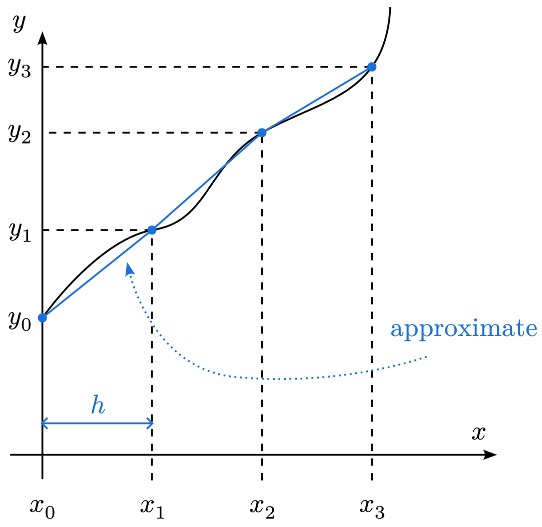
It is sometimes useful to consider functions evaluated at discrete points.

Consider again $5y' - 3y = 0$ with $y(0) = y_0$.

We can approximate this equation by discrete form at $\{x_n\}$ with $x_n = nh$ and $x_0 = 0$. With $y_n = y(x_n)$, we have

$$\left.\frac{dy}{dx}\right|_{x_0} \approx \frac{y_{n+1} - y_n}{h}.$$

[This is called the Forward Euler scheme, which is not a great approximation numerically.]



Substituting this approximation into the original equation gives

$$5\frac{y_{n+1}-y_n}{h} - 3y_n = 0$$

$$y_{n+1} = \left(1+\frac{3}{5}h\right)y_n \quad \text{(recurrence relation)}$$

Hence we have

$$\begin{aligned} y_n &= \left(1+\frac{3}{5}h\right)y_{n-1} \\ &= \left(1+\frac{3}{5}h\right)^2 y_{n-2} \\ &= \dots \\ &= \left(1+\frac{3}{5}h\right)^n y_0 \\ &= \left(1+\frac{3x}{5n}\right)^n y_0 \quad \text{by taking } h = \frac{x}{n}. \end{aligned}$$

Now take $x_n = x$ [this represents n steps from $x = 0$ to x] as $n \to \infty$. Then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_0\left(1+\frac{3x}{5n}\right)^n = y_0 \exp\left(\frac{3}{5}x\right)$$

which, thankfully, agrees with the continuous case.

4.2.3 Series Solutions

This is a powerful way to solve ODEs. Essentially, we are looking for soluions in the form of a power series,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

where we will determine a_n by substituting into the ODE.

Example 4.13

We shall get back to the example

$$5y' - 3y = 0.$$

Then, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} a_n n x^{n-1}.$$

Thus,

$$x \frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=1}^{\infty} a_n n x^n.$$

Also, by multiplying our original series by x,

$$\begin{aligned} xy &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=1}^{\infty} a_{n-1} x^n. \end{aligned}$$

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Therefore, substituting into the ODE gives

$$\sum_{n=1}^{\infty} \underbrace{\left(5a_n n - 3a_{n-1}\right)}_{\text{must vanish for all } n \geqslant 1} x^n = 0.$$

Hence we can derive the recurrence relation

$$a_n = \frac{3}{5n} a_{n-1}$$

Therefore

$$\begin{aligned} a_n &= \frac{3}{5n} a_{n-1} \\ &= \frac{3}{5n} \frac{3}{5(n-1)} a_{n-2} \\ &= \dots \\ &= \frac{3^n}{5^n n!} a_0. \end{aligned}$$

and

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{3x}{5}\right)^n = a_0 \exp\left(\frac{3x}{5}\right)$$

4.3 Forced (non-homogeneous) ODEs

In a forced ODE, there are terms not involving the dependent variable or its derivatives. In this case, in general, $y = 0$ is no longer a solution. To solve these equations, we should do the followings.

1. Find **any** solution of the forced equation, called a particular integral (PI) $y_p(x)$.
2. Write the general solution as $y(x) = y_p(x) + y_c(x)$, and find the complementary function (CF) $y_c(x)$ that satisfies the corresponding homogeneous equation.
3. Combine y_p and y_c to get the general solution.

This method is general for **linear** ODEs.

4.3.1 Constant Forcing

Example 4.14 (Constant forcing)

Consider

$$5y' - 3y = \underbrace{10}_{\text{constant forcing term}}.$$

A particular integral is $y_{p(x)} = -\frac{10}{3}$, since substituting it gives

$$5\frac{dy_p}{dx} - 3y_p = 0 - 10 = 10.$$

Then the complementary function is the solution of the homogeneous equation $5y' - 3y = 0$, which we have already solved as $y_c(x) = A\exp\left(\frac{3x}{5}\right)$.

Hence, the general solution is

$$y(x) = A\exp\left(\frac{3x}{5}\right) - \frac{10}{3}.$$

4.3.2 Eigenfunction Forcing

The forcing term may also be an eigenfunction of the differential operator.

Example 4.15 (Radioactive Decay)

Consider the decay between three isotopes A, B, C, with decay constants k_a, k_b for A and B respectively.

$$\underbrace{\mathrm{A}}_{a(t)} \xrightarrow{k_a a} \underbrace{\mathrm{B}}_{b(t)} \xrightarrow{k_b b} \underbrace{\mathrm{C}}_{c(t)}$$

Thus we have

$$\frac{da}{dt} = -k_a a$$
$$a = a_0 \exp(-k_a t)$$

and also

$$\frac{db}{dt} = k_a a - k_b b$$

$$\frac{db}{dt} + k_b b = \underbrace{k_a a_0 \exp(-k_a t)}_{\text{forcing term is an eigenfunction}}.$$

We shall try the particular integral of the form $b_p(t) = \beta \exp(-k_a t)$. Substituting it gives

$$-k_a \beta \exp(-k_a t) + k_b \beta \exp(-k_a t) = k_a a_0 \exp(-k_a t)$$

$$(k_b - k_a)\beta = k_a a_0$$

$$\beta = \frac{k_a a_0}{k_b - k_a} \quad (k_a \neq k_b).$$

> ***Remark.*** If $k_a = k_B$, we need another particular integral. See Example 4.17.

Hence $b_c(t)$ is the solution of the homogeneous equation $\frac{db_c}{dt} + k_b b_c = 0$. Thus

$$b_c(t) = D \exp(-k_b t).$$

Thus, the general solution is

$$b(t) = \frac{k_a a_0}{k_b - k_a} \exp(-k_a t) + D \exp(-k_b t).$$

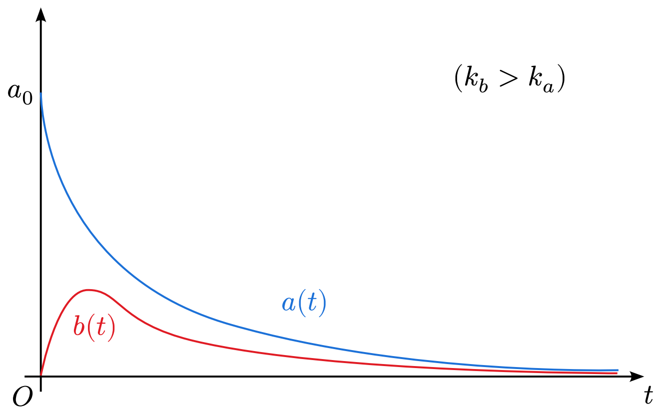
Now, if we were given the initial conditions $b(0) = 0$, then

$$b(0) = \frac{k_a a_0}{k_b - k_a} + D = 0$$

$$D = -\frac{k_a a_0}{k_b - k_a}.$$

Hence,

$$b(t) = \frac{k_a a_0}{k_b - k_a}[\exp(-k_a t) - \exp(-k_b t)].$$



4.4 Non-constant Coefficients

The general form of such equations is

$$a(x)\frac{\mathrm{d}y}{\mathrm{d}x} + b(x)y = c(x).$$

We can get the standard form by dividing both sides by $a(x)$ (assuming $a(x) \neq 0$):

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x).$$

To solve these equations, we use an integrating factor (IF) $\mu(x)$. Multiplying our standard form by $\mu(x)$ gives

$$\mu y' + \mu P y = \mu f.$$

We want the left-hand side to be $(\mu y)'$, so we require $\mu' = \mu P$ by the product rule. This is a separable equation, so

$$\frac{\mu'}{\mu} = P \Rightarrow \int P\,\mathrm{d}x = \int \frac{\mu'}{\mu}\,\mathrm{d}x = \ln u. \quad \text{(up to constant)}$$

Therefore,

$$\mu(x) = \exp\left(\int^x P(u)\,\mathrm{d}u\right),$$

which is unique up to an irrelevant constant factor. Hence, the original equation becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu f \\ \Rightarrow \mu y &= \int \mu f\,\mathrm{d}x.\end{aligned}$$

Example 4.16

Consider the equation

$$\begin{aligned} xy' + (1-x)y &= 1 \\ y' + \underbrace{\frac{1-x}{x}}_{P(x)} y &= \underbrace{\frac{1}{x}}_{f(x)}. \end{aligned}$$

The integrating factor is

$$\begin{aligned}\mu(x) &= \exp\left(\int \frac{1-x}{x}\,\mathrm{d}x\right) \\ &= \exp\left(\int \frac{1}{x}\,\mathrm{d}x - \int 1\,\mathrm{d}x\right) \\ &= \exp(\ln x - x) = xe^{-x}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dx}(xe^{-x}y) &= e^{-x}\\ xe^{-x}y &= -e^{-x}+C\\ y &= \frac{Ce^{x}-1}{x}.\end{aligned}$$

Remark. We require $c = 1$ if we want y to be finite as $x \to 0$.

Let us get back to Example 4.15.

Example 4.17 (Radioactive Decay, Revisited)

We have

$$\frac{db}{dt} + k_b b = k_a a_0 \exp(-k_a t).$$

We can identify $P(t) = k_b$ and $f(t) = k_a a_0 \exp(-k_a t)$. Thus, the integrating factor is

$$\begin{aligned}\mu(t) &= \exp\left(\int k_b \, dt\right)\\ &= \exp(k_b t).\end{aligned}$$

Hence,

$$\frac{d}{dt}(\exp(k_b t)b) = k_a a_0 \exp((k_b - k_a)t).$$

Let us consider two cases.

- If $k_b \neq k_a$, then

$$\begin{aligned}\exp(k_b t)b &= \frac{k_a a_0}{k_b - k_a}\exp((k_b - k_a)t) + C\\ b &= \frac{k_a a_0}{k_b - k_a}\exp(-k_a t) + C\exp(-k_b t).\end{aligned}$$

  This is exactly the solution we arrived at in Example 4.15, using the PI and CF method.

- If $k_b = k_a$, then

$$\frac{d}{dt}(\exp(k_a t)b) = k_a a_0$$

  Thus,

$$\begin{aligned}\exp(k_a t)b &= k_a a_0 t + C\\ b &= k_a a_0 t\exp(-k_a t) + C\exp(-k_a t).\end{aligned}$$

  Note that the particular integral is now proportional to $t\exp(-k_a t)$, which is different from the previous case. This is called the *resonance case*.

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5 Nonlinear First-Order ODEs

Recall that a non-linear ODE is one in which the dependent variable [usually y] and its derivatives appear with exponents other than 1 or are multiplied together.

The general form of a first-order nonlinear ODE is

$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0. \tag{«1}$$

[The term in $\frac{dy}{dx}$ could be nonlinear, but it is not considered here.]

We have two special cases of nonlinear first-order ODEs that we can solve: separable equations and exact equations.

5.1 Separable Equations

Definition 5.1 (Separable Equations)

A first-order ODE is called **separable** if it can be written in the form

$$q(y)\,dy = p(x)\,dx.$$

These equations can be solved by integrating both sides.

$$\int q(y)\,dy = \int p(x)\,dx.$$

Example 5.2

Consider the equation

$$\left(x^2y - 3y\right)\frac{dy}{dx} - 2xy^2 = 4x.$$

Rearranging the equation gives

$$\begin{aligned} y\left(x^2-3\right)\frac{dy}{dx} &= 2x\left(2+y^2\right) \\ \frac{y}{2+y^2}\,dy &= \frac{2x}{x^2-3}\,dx. \\ \frac{1}{2}\ln\left|2+y^2\right| &= \ln\left|x^2-3\right| + C \\ \left|2+y^2\right|^{\frac{1}{2}} &= A\left|x^2-3\right|. \end{aligned}$$

5.2 Exact equations

Definition 5.3 (Exact Equations)

The ODE Equation 1 is called **exact** if $P(x, y)\,dx + Q(x, y)\,dy$ is an exact differential, *i.e.* there exists a function $f(x, y)$ such that

$$df = P(x, y)\,dx + Q(x, y)\,dy.$$

In particular, if Equation 1 is exact, then $df = 0$ and $f(x, y)$ being a constant is a solution.

If an ODE is exact, then using the multivariate chain rule, we have

$$df = \underbrace{\frac{\partial f}{\partial x}}_{P} dx + \underbrace{\frac{\partial f}{\partial y}}_{Q} dy.$$

So, we can solve exact equations by finding a function $f(x, y)$ such that $\frac{\partial f}{\partial x} = P(x, y)$ and $\frac{\partial f}{\partial y} = Q(x, y)$.

Since partial derivatives commute, we have

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

This is a necessary condition but not sufficient for exactness:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem 5.4

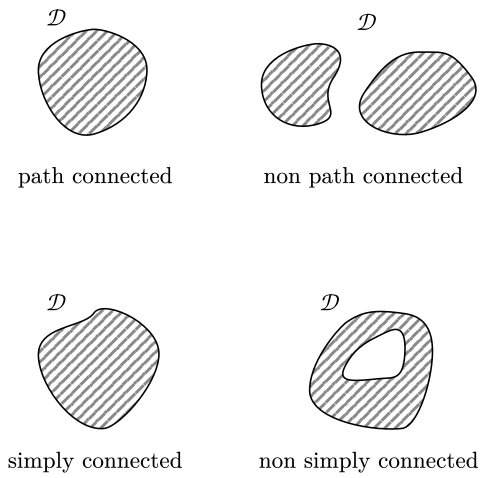
If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout a simply connected domain $\mathcal{D}$, then $P\,dx + Q\,dy$ is an exact differential of a single-valued function $f(x, y)$ on $\mathcal{D}$.

Definition 5.5 (Simply Connected Domain)

A domain $\mathcal{D}$ is **simply connected** if it is path connected, any any closed curve can be continuously shrunk to a point within $\mathcal{D}$ without leaving $\mathcal{D}$.

Definition 5.6 (Path Connectedness)

A domain $\mathcal{D}$ is **path connected** if every pair of points in $\mathcal{D}$ can be connected by a path in $\mathcal{D}$.



Example 5.7

Consider the equation

$$6y(y-x)\frac{dy}{dx}+2x-3y^2=0.$$

Rewriting gives

$$\underbrace{\left(2x-3y^2\right)}_{P}dx+\underbrace{6y(y-x)}_{Q}dy=0$$

Hence we have

$$\frac{\partial P}{\partial y}=-6y=\frac{\partial Q}{\partial x}=-6y.$$

Hence, the equation is exact in any simply connected domain.

Thus

$$P=\left.\frac{\partial f}{\partial x}\right|_y=2x-3y^2$$
$$f(x,y)=x^2-3xy^2+h(y)$$

and similarly,

$$Q=6y^2-6xy=\left.\frac{\partial f}{\partial y}\right|_x=-6xy+\frac{dh}{dy}$$
$$\frac{dh}{dy}=6y^2$$
$$h(y)=2y^3+C.$$

Hence $f(x,y)=x^2-3xy^2+2y^3+C=$ constant, and we have the implicit solution

$$x^2-3xy^2+2y^3=A.$$

5.3 Solution Curves and Isoclines

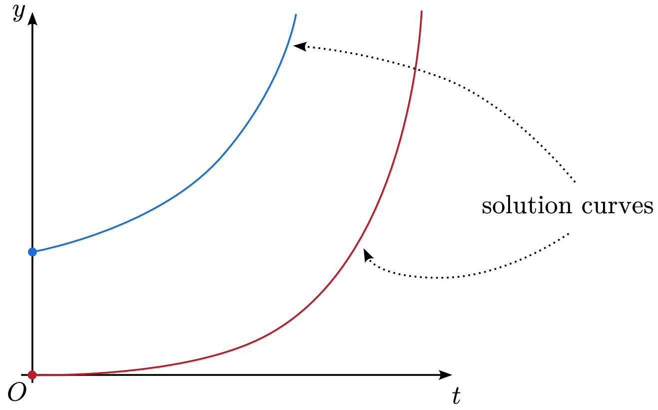
The general idea is that nonlinear ODEs are often impossible to solve in simple closed forms. We can still analyse the behaviour of solutions using graphical methods.

5.3.1 Solution Curves

Consider

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y). \quad (y = y(t))$$

Then, each initial condition (*e.g.* $y(0) = y_0$) generates a distinct solution curve (trajectory).



We can still sketch these solution curves without actually solving the ODE.

Example 5.8

We can solve the following equation to illustrate the ideas:

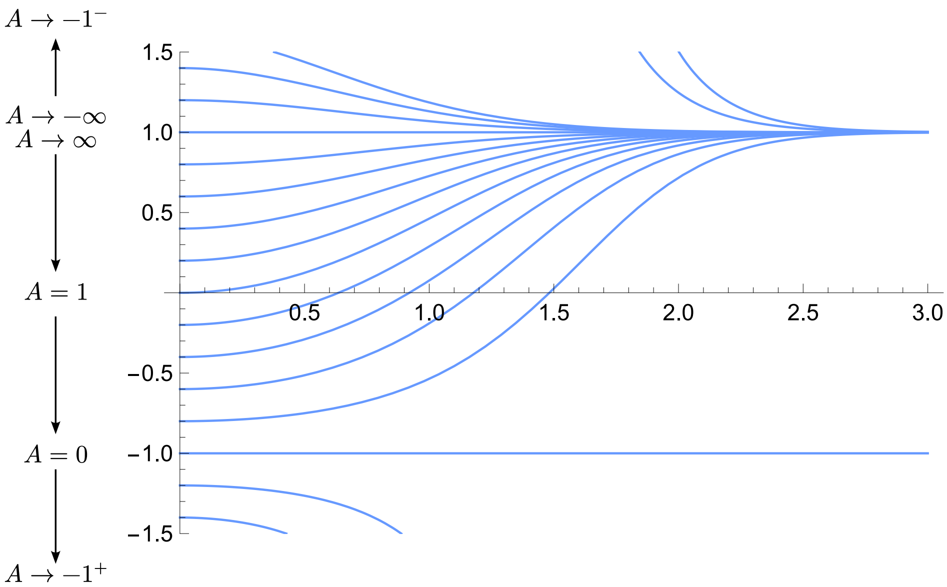
$$\frac{\mathrm{d}y}{\mathrm{d}t} = t\left(1 - y^2\right).$$

This is a separable equation:

$$\begin{aligned}
\frac{1}{1-y^2}\,\mathrm{d}y &= t\,\mathrm{d}t \\
\frac{1}{2}\left(\frac{1}{1+y} + \frac{1}{1-y}\right)\mathrm{d}y &= t\,\mathrm{d}t \\
\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| &= \frac{1}{2}t^2 + C \\
&\vdots \\
y &= \frac{A - \mathrm{e}^{-t^2}}{A + \mathrm{e}^{-t^2}} \quad \text{where } A = \begin{cases} \mathrm{e}^{2C} & \text{if } |y| < 1 \\ -\mathrm{e}^{2C} & \text{if } |y| > 1 \end{cases}.
\end{aligned}$$

Hence, we have a family of solution curves parameterised by A.

If $y(0) = y_0$, then $A = \frac{1+y_0}{1-y_0}$. Now, we can sketch the solution curves for various y_0.



Note that, some general behavior follows directly from the ODE:

- $\dot{y} = 0$ for all t if $y = \pm 1$. Hence we have two constant solution curves at $y = 1$ and $y = -1$.
- $\dot{y} = 0$ at $t = 0$ for any y. Hence, all solution curves have a horizontal tangent at $t = 0$.

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The equation $\frac{dy}{dt} = f(t, y)$ gives us the gradient of the solution through (t, y).

Note that the solution curves can't cross if $f(t, y)$ is single valued.

Definition 5.9 (Slope field)

The **slope field** represents the gradient field by short *sticks*, one centered on each point. It is tangent to the solution curves.

Example 5.10

Consider

$$\frac{dy}{dt} = t\left(1 - y^2\right).$$

Then, for $t > 0$,

- $\dot{y} > 0$ for $|y| < 1$ and
- $\dot{y} < 0$ for $|y| > 1$.

Definition 5.11 (Isoclines)

Isoclines are curves along which $\dot{y}$ is a constant. This can sometimes be useful when sketching solution curves.

Example 5.12

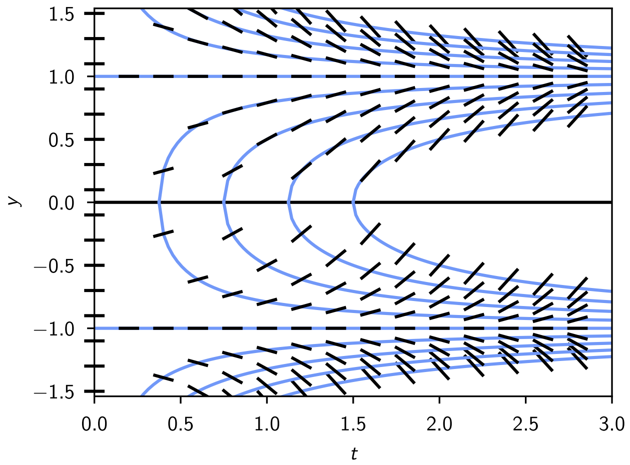
For

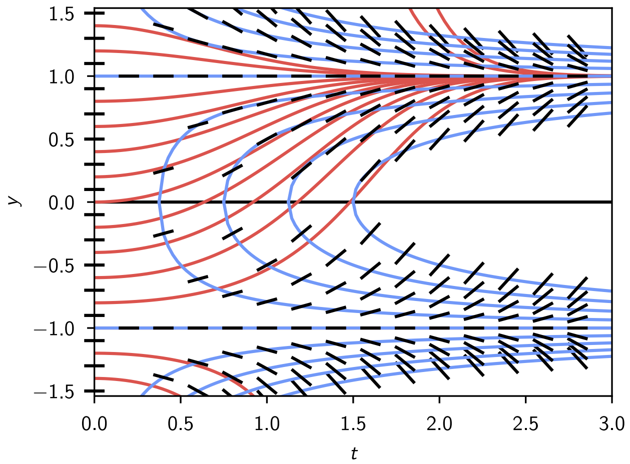
$$\frac{\mathrm{d}y}{\mathrm{d}t} = t\left(1 - y^2\right),$$

we have

$$t\left(1 - y^2\right) = D$$
$$y^2 = 1 - \frac{D}{t}$$

for some constant D.





5.4 Fixed (Equilibrium) Points and Stability

These points often reveal important features of ODEs.

Definition 5.13 (Fixed Point)

A **fixed point** (or **equilibrium point**) of an ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$ is a constant solution $y = c$, such that $\frac{\mathrm{d}y}{\mathrm{d}t} = 0\ \forall t$.

Example 5.14

Consider

$$\frac{dy}{dt} = t\left(1 - y^2\right).$$

The fixed points are $y = \pm 1$. They have very different character as seen from the sketchs above. The solution curves $y = +1$ as $t \to \infty$, while those near $y = -1$ diverge away from it.

Definition 5.15 (Stability of Fixed Points)

A fixed point $y = c$ is

- **stable** if whenever y deviates slightly from c, y converges to c as $t \to \infty$.
- **unstable** if whenever y deviates slightly from c, y diverges from c as $t \to \infty$.

5.4.1 Perturbation Analysis and Stability

Let $y = c$ be a fixed point of $\frac{dy}{dt} = f(t, y)$ [*i.e.* $f(t, c) = 0$ for all t]. Consider a small perturbation about the fixed point:

$$y(t) = c + \varepsilon(t), \quad |\varepsilon| \ll 1.$$

We want to analyze how $\varepsilon(t)$ behaves as $t \to \infty$.

Then

$$\begin{aligned}\frac{dy}{dt} = \frac{d\varepsilon}{dt} &= f(t, c + \varepsilon) \\ &= \underbrace{f(t, c)}_{=0 \text{ at F.P.}} + \varepsilon \frac{\partial f}{\partial y}(t, c) + O\left(\varepsilon^2\right)\end{aligned}$$

Linearize for small epsilon:

$$\frac{d\varepsilon}{dt} \approx \varepsilon \frac{\partial f}{\partial y}(t, c).$$

Not that this is a linear ODE in ε.

Note that if $\frac{\partial f}{\partial y}(t, c) = 0$, then we need higher order terms to determine stability.

Example 5.16

Consider the equation

$$\frac{dy}{dt} = t\left(1 - y^2\right).$$

We are aware that the fixed points are $y = \pm 1$. We have

$$\frac{\partial f}{\partial y} = -2ty = \begin{cases} -2t & y = +1 \\ +2t & y = -1 \end{cases}$$

Near $y = 1$:

$$\frac{d\varepsilon}{dt} \approx -2t\varepsilon \Rightarrow \varepsilon(t) = \varepsilon_0 e^{-t^2}.$$

Hence, as $t \to \infty$, $\varepsilon(t) \to 0$ and the fixed point at $y = 1$ is stable.

Near $y = -1$:

$$\frac{d\varepsilon}{dt} \approx 2t\varepsilon \Rightarrow \varepsilon(t) = \varepsilon_0 e^{t^2}.$$

Hence, as $t \to \infty$, $\varepsilon(t) \to \infty$ and the fixed point at $y = -1$ is unstable.

Remark. This perturbation analysis only work for small ε. Hence, we only really have the behavior of solutions close to the fixed point.

5.4.2 Autonomous Systems and Phase Portraits

Definition 5.17 (Autonomous system)

Autonomous systems are ODEs in which the independent variable (*e.g.* t) does not appear explicitly in the equation. *e.g.*

$$\frac{dy}{dt} = f(y).$$

First of all, these autonomous systems are separable. We can written

$$\int^{y} \frac{du}{f(u)} = t - t_0.$$

Hence, if $y(t)$ is a solution, then so is $y(t - t_0)$ for any constant t_0.

Although this equation is separable, we may not be able to solve it in closed form. Consider, near a fixed point $y = c$, we have

$$\frac{dy}{dt} = \frac{d\varepsilon}{dt} = \varepsilon \underbrace{\frac{df}{dy}(c)}_{\text{constant } k} = k\varepsilon$$
$$\varepsilon(t) = \varepsilon_0 e^{kt}.$$

Remark. $\frac{df}{dy}(c)$ is a constant since we are evaluating at some fixed point.

Hence, if

- $k = f'(c) < 0$, then the fixed point is stable.
- $k = f'(c) > 0$, then the fixed point is unstable.

Example 5.18 (Chemical kinetics)

Consider a chemical reaction

$$A + B \to C + D.$$

The number of molecules of A, B, C, D at time t are $a(t), b(t), c(t), d(t)$ respectively.

The initial numbers are $a(0) = a_0, b(0) = b_0, c(0) = 0, d(0) = 0$. Hence, we have the conservation laws

$$\begin{aligned} a(t) &= a_0 - c(t) \\ b(t) &= b_0 - c(t) \\ d(t) &= c(t) \end{aligned}$$

since one of A and B is consumed to produce one of C and D. Assume that the rate of reaction is proportional to the product of the numbers of A and B molecules (*e.g.* we considering dilute gases):

$$\frac{\mathrm{dc}}{\mathrm{dt}} = \lambda a(t) b(t) = \underbrace{\lambda(a_0 - c)(b_0 - c)}_{f(c)}.$$

Therefore, we have an example of an autonomous non-linear first-order ODE.

Note that the fixed points are $c = a_0$ and $c = b_0$ (corresponding to the complete consumption of either A or B).

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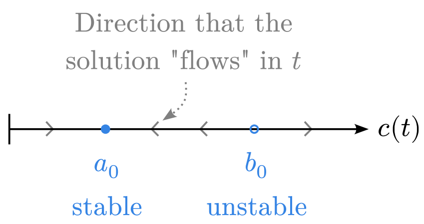
Now assume $a_0 < b_0$. Then, $c = b_0$ is unphysical. We shall now carry out perturbation analysis to determine the stability of the fixed points.

$$\frac{\mathrm{df}}{\mathrm{dc}} = \lambda(2c - a_0 - b_0) = \begin{cases} \lambda(a_0 - b_0) \text{ at } c = a_0 \\ \lambda(b_0 - a_0) \text{ at } c = b_0 \end{cases}$$

For $a_0 < b_0$, $c = a_0$ is a stable fixed point, while $c = b_0$ is an unstable fixed point.

We can sketch a 1D phase portrait to visualise the behaviour of solutions.



Example 5.19 (Population dynamics (logistic equation))

Consider a population of size $y(t)$. We have

- birth rate: αy with $\alpha > 0$,
- death rate: $\beta y + \gamma y^2$, where
  - βy models isolated deaths
  - γy^2 models deaths due to overcrowding.

Thus we have the ODE

$$\frac{dy}{dt} = \alpha y - \beta y - \gamma y^2 = \underbrace{(\alpha - \beta)y - \gamma y^2}_{f(y)}.$$

To make things simpler, let $\lambda = \alpha - \beta$ and $\gamma = \frac{\lambda}{Y}$. Then,

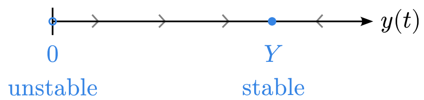
$$\frac{dy}{dt} = \lambda y\left(1 - \frac{y}{Y}\right) = f(y).$$

This is called a differential logistic equation. It is an example of an autonomous system. The fixed points are $y = 0$ and $y = Y$. We can carry out perturbation analysis to determine their stability.

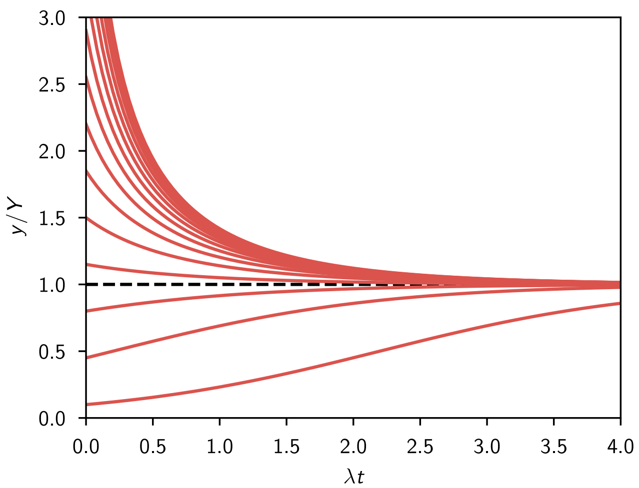
$$\frac{df}{dy} = \lambda\left(1 - \frac{2y}{Y}\right) = \begin{cases} \lambda & \text{at } y = 0 \\ -\lambda & \text{at } y = Y \end{cases}$$

For $\lambda > 0$ ($\alpha > \beta$), $y = 0$ is an unstable fixed point, while $y = Y$ is a stable fixed point.

We can sketch a 1D phase portrait to visualise the behaviour of solutions.



This equation can be solved exactly and sketched.



5.5 Fixed Points in Discrete Equations

Definition 5.20 (First order discrete equation)

A **first-order discrete equation** is a recurrence relation of the form

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots$$

Note that the RHS is independent of n, which is like an autonomous system in continuous ODEs.

Definition 5.21 (Fixed point of 1st order discrete equation)

A **fixed point** of a 1st order discrete equation is a value of x_n such that $x_{n+1} = x_n$, *i.e.* $x_n = f(x_n)$.

We can also analyse the stability of fixed points in discrete equations using perturbation analysis. Let x_f be a fixed point, and write $x_n = x_f + \varepsilon_n$, where $|\varepsilon_n| \ll 1$.

Then,

$$\begin{aligned} x_{n+1} &= x_f + \varepsilon_{n+1} \\ f(x_n) &= f(x_f + \varepsilon_n) \\ &= \underbrace{f(x_f)}_{=x_f \text{ at F.P.}} + \varepsilon_n f'(x_f) + O\left(\varepsilon_n^2\right) \\ \varepsilon_{n+1} &\approx \varepsilon_n f'(x_f). \end{aligned}$$

Therefore, x_f is

- **stable** if $|f'(x_f)| < 1$,
- **unstable** if $|f'(x_f)| > 1$.

Example 5.22

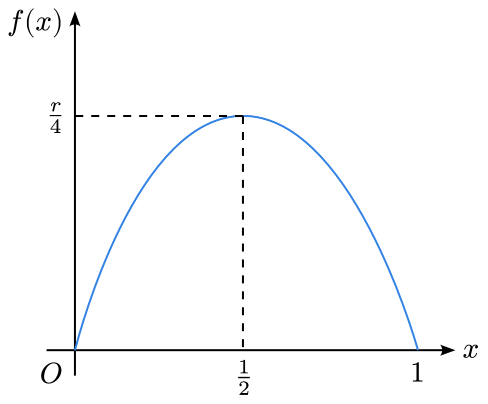
Consider the discrete equation

$$x_{n+1} = \underbrace{r x_n (1 - x_n)}_{f(x_n)}.$$

This is called a discrete logistic equation, or the logistic map. We can compare this with the continuous logistic equation in Example 5.19.

[This is useful to model population dynamics when we consider births at discrete time intervals only.]

We are only interested in $x_n \geqslant 0$. We can sketch the graph of $f(x)$ against x.



From the graph, if $0 \leqslant r \leqslant 4$, then $\{x_n\}$ stay within $[0, 1]$.

The fixed points satisfy $x_n = f(x_n) = r x_n (1 - x_n)$. Hence, the fixed points are

$$x_n = 0 \quad \text{and} \quad x_n = 1 - \frac{1}{r}.$$

However, be aware that the second fixed point only makes sense for $r > 1$ (in order to be non-negative).

We can carry out perturbation analysis to determine their stability.

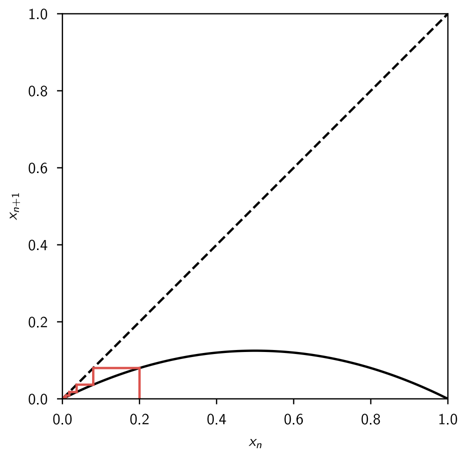
$$f'(x_n) = r - 2rx_n = \begin{cases} r & \text{at } x_n = 0 \\ 2 - r & \text{at } x_n = 1 - \frac{1}{r} \end{cases}$$

Therefore,

- For the fixed point at $x_n = 0$:
  - stable if $0 < r < 1$,
  - unstable if $r > 1$.
- For the fixed point at $x_n = 1 - \frac{1}{r}$:
  - stable if $1 < r < 3$,
  - unstable if $r > 3$.

We can illustrate the behaviour of solutions using cobweb diagrams.

- Consider $0 < r < 1$.



  This shows that $x = 0$ is a stable fixed point: solutions for any initial condition in $(0, 1)$ will converge to 0.

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6 Higher Order Linear ODEs

We shall focus on 2nd order linear ODEs, but many methods are also applicable to higher order linear ODEs.

Different to the 1st order case, closed form solutions to 2nd order linear ODEs don't always exist.

6.1 2nd Order ODEs with Constant Coefficients

The general form of a 2nd order linear ODE with constant coefficients is

$$\underbrace{a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy}_{\mathcal{D}(y)}=f(x)$$

where a, b, c are constants and $f(x)$ is a given function, and $\mathcal{D}$ is the differential operator

$$\mathcal{D}\equiv a\frac{\mathrm{d}^2}{\mathrm{d}x^2}+b\frac{\mathrm{d}}{\mathrm{d}x}+c$$

which is linear.

Definition 6.1 (Linear differential operator)

A differential operator $\mathcal{D}$ is **linear** if for any functions $y_1(x), y_2(x)$ and constants $\alpha, \beta \in \mathbb{R}$,

$$\mathcal{D}(\alpha y_1+\beta y_2)=\alpha\mathcal{D}y_1+\beta\mathcal{D}y_2,$$

which is called the principle of superposition.

We can exploit the linearity of $\mathcal{D}$ to solve 2nd order linear ODEs.

1. Find the complementary functions that satisfy the homogeneous equation:
$$\mathcal{D}(y_c)=0.$$
2. Find a particular integral y_p that satisfies the non-homogenous equation:
$$\mathcal{D}(y_p)=f(x).$$
3. A solution of the full equation is then given by
$$y=y_c+y_p,$$
since by linearity, $\mathcal{D}(y_c+y_p)=\mathcal{D}(y_c)+\mathcal{D}(y_p)=0+f(x)=f(x)$.

A 2nd order ODE has **two** linearly independent complementary functions, so the general solution to the full equation is

$$y(x)=C_1y_{c_1}(x)+C_2y_{c_2}(x)+y_p(x),$$

Definition 6.2 (Linearly dependent functions)

A set of N functions $\{f_i(x)\}$ is **linearly dependent** if there exist N constants c_i, not all zero, such that

$$\sum_{i=1}^{N} c_i f_i(x) = 0 \quad \forall x.$$

Definition 6.3 (Linearly independent functions)

A set of functions is **linearly independent** if it is not linearly dependent.

One can compare the two definitions above with the definition of vectors.

> ***Remark.*** Equivalently, if one or more of the functions $f_i(x)$ can be written as a linear combination of the others, then the set is linearly dependent.

6.2 Complementary Functions

Recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\lambda x} = \lambda \mathrm{e}^{\lambda x}$$

so $\mathrm{e}^{\lambda x}$ is also an eigenfunction of $\mathcal{D}$, where

$$\mathcal{D}\left(\mathrm{e}^{\lambda x}\right) = \underbrace{\left(a\lambda^2 + b\lambda + c\right)}_{\text{eigenvalue}}\mathrm{e}^{\lambda x}.$$

The complementary functions satisfy $\mathcal{D}y_c = 0$, *i.e.* y_n are eigenfunctions of $\mathcal{D}$ with eigenvalue 0.

> ***Remark.*** We can prove that the eigenfunctions of a linear differential operator with constant coefficients are of the form $\mathrm{e}^{\lambda x}$.

Therefore, $y_c = A\mathrm{e}^{\lambda x}$ with λ satisfying the **characteristic equation** of $\mathcal{D}$:

$$a\lambda^2 + b\lambda + c = 0.$$

There are two roots, λ_1, λ_2, to the characteristic equation, which can be real or complex.

- **Case 1: $\lambda_1 \neq \lambda_2$**

  We have two linearly independent complementary functions:

  $$y_{c_1} \propto \mathrm{e}^{\lambda_1 x}, \quad y_{c_2} \propto \mathrm{e}^{\lambda_2 x}.$$

  Hence, the most general complementary function is

  $$y_c = C_1 y_{c_1}(x) + C_2 y_{c_2}(x).$$

  In the language of linear algebra, the space of complementary functions is a 2-dimensional vector space spanned by the basis $\left\{y_{c_1}, y_{c_2}\right\}$.

  > ***Remark.*** The roots may be complex, which will lead to oscillations.

- **Case 2:** $\lambda_1 = \lambda_2$ (degenerate case)

  We only have one linearly independent complementary function $y_c \propto e^{\lambda_1 x}$. See the following examples to find the second complementary function.

Example 6.4 (Real, non-degenerate roots)

Consider the ODE

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

The characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0,$$

which has roots $\lambda_1 = 2, \lambda_2 = 3$.

Therefore, the general complementary function is

$$y_c(x) = Ae^{2x} + Be^{3x}.$$

where A, B are constants.

Example 6.5 (Complex, non-degenerate roots)

Consider the ODE

$$\frac{d^2y}{dx^2} + 4y = 0.$$

The characteristic equation is

$$\lambda^2 + 4 = 0,$$

which has roots $\lambda = \pm 2i$.

Therefore, the general complementary function is

$$y_c(x) = Ae^{2ix} + Be^{-2ix}.$$

Note that using Euler's formula, we can rewrite this as

$$y_c(x) = \alpha\cos(2x) + \beta\sin(2x),$$

where α, β are constants ($\alpha = A + B$, $\beta = i(A - B)$).

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Example 6.6 (Degenerate roots and "detuning")

Consider the ODE

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0,$$

which has a degenerate root $\lambda_1 = \lambda_2 = 2$.

We only have one complementary function $y_c \propto e^{2x}$.

To find the second complementary function, we can "detune" the equation slightly to remove the degeneracy, by considering a slightly modified equation:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + \left(4 - \varepsilon^2\right)y = 0,$$

where $\varepsilon \ll 1$.

Then the characteristic equation is

$$\lambda^2 - 4\lambda + \left(4 - \varepsilon^2\right) = 0,$$

which has roots $\lambda_1 = 2 + \varepsilon, \lambda_2 = 2 - \varepsilon$.

Hence, the general complementary function is

$$\begin{aligned} y_c &= Ae^{(2+\varepsilon)x} + Be^{(2-\varepsilon)x} \\ &= e^{2x}\left(Ae^{\varepsilon x} + Be^{-\varepsilon x}\right) \\ &= e^{2x}\left[(A+B) + \varepsilon(A-B)x + O\left(A\varepsilon^2\right) + O\left(B\varepsilon^2 x^2\right)\right] \quad \text{as } \varepsilon \to 0. \quad \text{(Taylor series)} \end{aligned}$$

Consider the initial conditions $y(0) = C$, $y'(0) = D$ to both the original and detuned equations. We have

$$\begin{aligned} A + B &= C \\ 2C + \underbrace{\varepsilon(A - B)}_{\substack{2A-C \\ \text{or } C-2B}} &= D \end{aligned}$$

$$\begin{cases} A = \frac{1}{2}\left(C + \frac{D-2C}{\varepsilon}\right) \\ B = \frac{1}{2}\left(C - \frac{D-2C}{\varepsilon}\right) \end{cases}$$

Therefore,

$$O\left(A\varepsilon^2 x^2\right) = O\left(\varepsilon x^2\right) \to 0 \quad \text{as } \varepsilon \to 0.$$

We have

$$\begin{aligned} \alpha &= A + B &&= O(1) \\ \beta &= \varepsilon(A - B) &&= O(1). \end{aligned}$$

Therefore, we can write the general complementary function as

$$y_c(x) = (A + B)e^{2x} + \varepsilon(A - B)xe^{2x} = \alpha e^{2x} + \beta xe^{2x}.$$

General rule. If $y_{c_1}(x)$ is a degenerate complementary function of linear ODE with constant coefficients, then

$$y_{c_2}(x) = x y_{c_1}(x)$$

is a second linearly independent complementary function.

6.3 Homogeneous 2nd Order ODEs with Non-Constant Coefficients

The general form of this kind of ODE is

$$y'' + p(x)y' + q(x)y = 0.$$

6.3.1 Second Complementary Function – Reduction of Order

We wish to find a 2nd solution $y_2(x)$ given one solution $y_1(x)$.

We will try from the form

$$y_2(x) = v(x)y_1(x)$$

where $v(x)$ is an unknown function to be determined. We have

$$\begin{aligned} y_2' &= v'y_1 + vy_1', \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

Substituting into the ODE, we get

$$v''y_1 + (2y_1' + py_1)v' + \underbrace{(y_1'' + py_1' + qy_1)}_{\substack{\text{same ODE for } y_1 \\ =0}} v = 0.$$

Therefore,

$$v''y_1 + (2y_1' + py_1)v' = 0.$$

Let $u = v'$, then

$$u'y_1 + (2y_1' + py_1)u = 0.$$

This is a separable 1st order ODE for u:

$$\frac{\mathrm{d}u}{u} = -\frac{2y_1'}{y_1} - p\,\mathrm{d}x.$$

Integrating both sides, we get

$$\begin{aligned} \ln|u| &= -2\ln y_1 - \int_0^x p(t)\,\mathrm{d}t + \ln A \\ u(x) &= \frac{A}{y_1^2}\exp\left[-\int_0^x p(t)\,\mathrm{d}t\right]. \end{aligned}$$

Hence, we can integrate u to find v, and then find y_2.

We shall see an example with constant coefficients first.

Example 6.7

Consider the ODE

$$y'' - 4y' + 4y = 0$$

where we have $p(x) = -4$, $q(x) = 4$.

One solution is $y_1(x) = e^{2x}$. Therefore,

$$\frac{u'}{u} = -4 + 4 = 0$$
$$u = \text{constant}$$

Hence, $v' = \text{constant}$, so $v(x) = Ax + B$.

Therefore, the second solution is

$$y_2(x) = v(x)y_1(x) = (Ax + B)e^{2x}.$$

Hence xe^{2x} is a second linearly independent solution.

6.3.2 Phase Space

For an nth order linear ODE

$$y^n + p(x)y^{n-1} + \dots + q(x)y = f(x).$$

$y^n(x)$ is determined by $y(x), y'(x), \dots, y^{n-1}(x)$, and higher derivatives are determined by differentiating the ODE.

Hence, we can construct Taylor series about x_0 if $y(x_0), \dots, y^{n-1}(x_0)$ are specified.

We say that the state of the system at any x is fully specified by an n-dimensional solution vector.

$$\mathbf{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{n-1}(x) \end{pmatrix}.$$

Remark.

- At any x, $\mathbf{Y}(x)$ defines a point in n-dimensional phase space.
- As x varies, $\mathbf{Y}(x)$ traces out a trajectory in phase space.

Example 6.8

Consider the 2nd order linear ODE

$$y'' + 4y = 0.$$

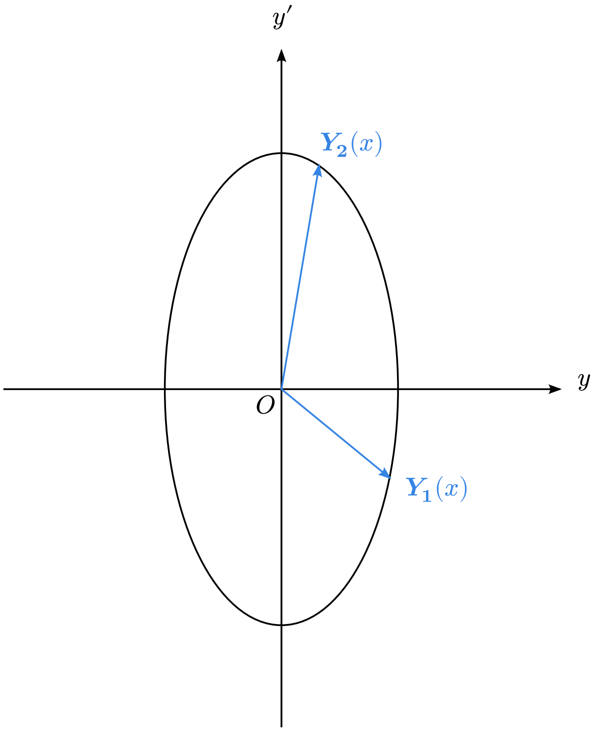
We know that the solutions are

$$y_1(x) = \cos 2x, \quad y_2(x) = \sin 2x.$$

Hence the solution vectors are

$$\mathbf{Y}_1(x) = \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix}$$
$$\mathbf{Y}_2(x) = \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix}.$$

In this case we have a 2D phase space.



Note that $\mathbf{Y}_1(x)$ and $\mathbf{Y}_2(x)$ are linearly independent vectors, so we can use them as a basis for the phase space.

6.3.3 Wronskian and Linear Dependence

Recall that $\{y_i(x)\}$ are linearly dependent if there exist constants c_i, not all zero, such that

$$\sum_{i=1}^{N} c_i y_i(x) = 0 \quad \forall x.$$

Hence, we can differentiate this equation $N-1$ times to get

$$\sum_{i=1}^{N} c_i \mathbf{Y}_i(x) = \mathbf{0} \quad \forall x.$$

so $\{y_i\}$ being linearly dependent implies that $\{\mathbf{Y}_i\}$ is linearly dependent.

Definition 6.9 (Fundamental matrix)

Given n solution vectors $\{\mathbf{Y}_i(x)\}$ of an nth order linear ODE, the **fundamental matrix** is the $n \times n$ matrix whose columns are the solution vectors:

$$\begin{pmatrix} | & | & & | \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_n \\ | & | & & | \end{pmatrix}$$

Definition 6.10 (Wronskian)

The **Wronskian** of the functions $\{y_i(x)\}$ is defined as the determinant of the fundamental matrix:

$$\begin{aligned} W(x) &= \det\left[\begin{pmatrix} | & | & & | \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_n \\ | & | & & | \end{pmatrix}\right] \\ &= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{vmatrix}. \end{aligned}$$

From the above, if $\{y_i(x)\}$ are linearly dependent, then $W(x) = 0$ for all x.

It follows that if $W(x_0) \neq 0$ for some x_0, then $\{y_i(x)\}$ are linearly independent.

Important. $W(x) = 0$ for all x does not necessarily imply that $\{y_i(x)\}$ are linearly dependent.

Example 6.11

Consider the ODE

$$y'' + 4y = 0.$$

The Wronskian of the two solutions $y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$ is

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} \\ &= 2\left(\cos^2 2x + \sin^2 2x\right) = 2. \end{aligned}$$

Hence the two solutions are linearly independent.

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6.3.4 Abel's Theorem

Theorem 6.12 (Abel's Theorem)

Given any 2 solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

if $p(x)$ and $q(x)$ are continuous on an interval I, then either the Wronskian $W(x) = 0$ for all $x \in I$ or $W(x) \neq 0$ for all $x \in I$.

Sketch Proof. We have

$$\begin{aligned} W &= y_1y_2' - y_2y_1' \\ W' &= y_1y_2'' + y_1'y_2' - y_2'y_1' - y_2y_1'' \\ &= y_1y_2'' - y_2y_1'' \\ &= -y_1(py_{2'} + qy_2) + y_2(py_{1'} + qy_1) \\ &= -pW. \end{aligned}$$

This is a separable ODE for W:

$$\begin{aligned} \frac{\mathrm{d}W}{W} &= -p(x)\,\mathrm{d}x \\ W &= W(x_0)\underbrace{\exp\left[-\int_{x_0}^{x} p(u)\,\mathrm{d}u\right]}_{\text{never zero}} \quad \text{(Abel's identity).} \end{aligned}$$

Hence, if $W(x_0) = 0$, then $W(x) = 0$ for all $x \in I$; otherwise, $W(x) \neq 0$ for all $x \in I$.

The geometric interpretation is that the solution vectors are either always collinear or never collinear in the phase space.

Corollary 6.13

If $p(x) = 0$, then the Wronskian is constant.

We can find $W(x)$ without knowing the solutions explicitly.

Example 6.14 (Bessel's equation)

Consider

$$\begin{aligned} x^2y'' + xy' + \left(x^2 - n^2\right)y &= 0 \\ y'' + \underbrace{\frac{1}{x}}_{p()}y' + \left(1 - \frac{n^2}{x^2}\right)y &= 0. \end{aligned}$$

Then, Abel's identity gives

$$W(x) = W(x_0)\exp\left[-\int_{x_0}^{x} \frac{1}{t}\,\mathrm{d}t\right] = W(x_0)\frac{x_0}{x}.$$

Application. Abel's identity can be usde to find a second solution y_2 given a known solution y_1. Consider that we have

$$y_1 y_2' - y_2 y_1' = W(x_0) \exp\left[-\int_{x_0}^{x} p(u)\, du\right].$$

Dividing both sides by y_1^2, we get

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{W(x_0)}{y_1^2} \exp\left[-\int_{x_0}^{x} p(u)\, du\right].$$

This is the same as we had using reduction of order.

Abel's theorem can be generalised.

Theorem 6.15 (Generalisation of Abel's Theorem)

Abel's theorem holds for solutions of nth order homogeneous linear ODEs.

6.4 Linear Equidimensional ODEs

These ODEs are related to ODEs with constant coefficients.

Definition 6.16 (Linear 2nd order equidimensional ODE)

A linear 2nd order ODE is **equidimensional** if it is of the form

$$ax^2 y'' + bxy' + cy = f(x),$$

where a, b, c are constants.

Proposition 6.17 (Scaling property of solutions of the homogeneous equation)

If $g(x)$ is a solution of the homogeneous equidimensional ODE with $f(x) = 0$, then so is $y = g(\alpha x)$ for any constant α.

Proof. We have

$$\frac{dg(\alpha x)}{dx} = \frac{dg(\alpha x)}{d(\alpha x)\frac{d(\alpha x)}{dx}} = g'(\alpha x)\alpha$$

$$x\frac{dy}{dx} = (\alpha x) g'(\alpha x)$$

$$x^2 \frac{d^2 y}{dx^2} = (\alpha x)^2 g''(\alpha x).$$

Therefore,

$$
\begin{aligned}
ax^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+bx\frac{\mathrm{d}y}{\mathrm{d}x}+cy &= a(\alpha x)^2g''(\alpha x)+b(\alpha x)g'(\alpha x)+cg(\alpha x)\\
&= au^2g''(u)+bug'(u)+cg(u).\\
&= 0 \quad \text{(since } g \text{ is a solution of the homogeneous equation)}
\end{aligned}
$$

6.4.1 Solving by Eigenfunctions

$$
x\frac{\mathrm{d}}{\mathrm{d}x}\left(x^k\right)=kx^k
$$

so x^k is an eigenfunction of the operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ with eigenvalue k.

It suggests that we should look for complementary functions of the form $y=x^k$. Substituting into the homogeneous equation, we get

$$
\begin{gathered}
x^k[ak(k-1)+bk+c]=0 \quad \text{for all } x\\
ak(k-1)+bk+c=0\\
ak^2+(b-a)k+c=0.
\end{gathered}
$$

Let the solutions be k_1, k_2. Then the complementary functions are

$$
y_c(x)=C_1x^{k_1}+C_2x^{k_2}.
$$

if $k_1\neq k_2$.

6.4.2 Solving by Substitution

Substitute $z=\ln x$, then

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}z}&=\frac{\mathrm{d}x}{\mathrm{d}z}\frac{\mathrm{d}y}{\mathrm{d}x}\\
\frac{\mathrm{d}^2y}{\mathrm{d}z^2}&=\underbrace{e^z\frac{\mathrm{d}y}{\mathrm{d}x}}_{x\frac{\mathrm{d}y}{\mathrm{d}x}}+\underbrace{e^{2z}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}_{x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}.
\end{aligned}
$$

Substituting into the ODE, we get

$$
\begin{gathered}
a\left(\frac{\mathrm{d}^2y}{\mathrm{d}z^2}-\frac{\mathrm{d}y}{\mathrm{d}z}\right)+b\frac{\mathrm{d}y}{\mathrm{d}z}+cy=f(e^z)\\
a\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+(b-a)\frac{\mathrm{d}y}{\mathrm{d}z}+cy=f(e^z).
\end{gathered}
$$

Now we have a linear ODE with constant coefficients in the variable z, which we can solve using the methods discussed earlier. The characteristic equation is

$$
a\lambda^2+(b-a)\lambda+c=0.
$$

This is the same characteristic equation as before when we tried the eigenfunction method. The complementary functions are therefore, if $k_1\neq k_2$,

$$y_c(x) = C_1 e^{k_1 z} + C_2 e^{k_2 z}$$
$$y_c(x) = C_1 x^{k_1} + C_2 x^{k_2}.$$

We can now deal with the degenerate case $k_1 = k_2$ similarly as before.

$$y_{c(x)} = Ae^{k_1 z} + Bze^{k_1 z}$$
$$y_{c(x)} = Ax^{k_1} + B(\ln x)x^{k_1}.$$

6.5 Inhomogeneous (Forced) 2nd Order ODEs

We will discuss the methods to find particular integrals.

6.5.1 Constant Coefficient ODEs

We have the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x).$$

Now, use the following ansatz for y_p depending on the form of $f(x)$:

$f(x)$	**Try y_p of the form**
e^{mx}	Ae^{mx}
$\cos(mx)$ or $\sin(mx)$	$A\cos(mx) + B\sin(mx)$
Polynomial of degree n	$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

We can determine the constants by substituting into the ODE. Since the ODE is linear, we can superpose terms.

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Example 6.18

Consider the ODE

$$y'' - 5y' + 6y = 2x + e^{4x}.$$

We shall try a particular integral of the form

$$y_p(x) = \underbrace{Ax + B}_{\text{for } 2x} + \underbrace{Ce^{4x}}_{\text{for } e^{4x}}.$$

Substituting into the ODE, we get

$$\begin{aligned} y_p'' - 5y_p' + 6y_p &= 2x + e^{4x} \\ &= 6(Ax + B) - 5A + 6Ce^{4x} - 20Ce^{4x} + 16Ce^{4x} \\ &= 6Ax + (6B - 5A) + 2Ce^{4x}. \end{aligned}$$

Equating coefficients, we have

$$\begin{aligned} 6A &= 2 \\ 6B - 5A &= 0 \\ 2C &= 1. \end{aligned}$$

Therefore, $A = \frac{1}{3}$, $B = \frac{5}{18}$, $C = \frac{1}{2}$.

The complmentary function, as discussed earlier, is

$$y_c(x) = \alpha e^{2x} + \beta e^{3x}.$$

Hence, the general solution is

$$y(x) = \alpha e^{2x} + \beta e^{3x} + \left(\frac{1}{3}\right)x + \left(\frac{5}{18}\right) + \left(\frac{1}{2}\right)e^{4x}$$

where α, β are constants.

If the forcing term $f(x)$ involves a term that is in a complementary function, a resonance case occurs. In this case, we carry out detuning.

Example 6.19 (Resonance)

Consider the ODE

$$\ddot{y} + \omega_0^2 y = \sin(\omega_0 x)$$

which represents a simple harmonic oscillator driven at its natural frequency ω_0. We say that this oscillator is driven resonantly. We have the comcomplementary functions

$$y_c(x) = A\cos(\omega_0 x) + B\sin(\omega_0 x).$$

Since $\sin(\omega_0 x)$ is already in the complementary function, we consider detuning by looking at the slightly modified equation

$$\ddot{y} + \omega_0^2 y = \sin(\omega x)$$

with $\omega \neq \omega_0$.

We try a particular integral of the form

$$y_p(x) = C\sin(\omega x) + D\cos(\omega x).$$

We can see that D must be zero since there is no $\cos(\omega x)$ term on the RHS. Substituting into the ODE, we get

$$\begin{aligned} -C\omega^2\sin(\omega x) + \omega_0^2 C\sin(\omega x) &= \sin(\omega x) \\ C\left(\omega_0^2 - \omega^2\right) &= 1 \\ C &= \frac{1}{\omega_0^2 - \omega^2}. \end{aligned}$$

Note that the limit $\omega \to \omega_0$ does not exist since C diverges. We can add in a complementary function to regularise the limit:

$$y_p(t) = \frac{1}{\omega_0^2 - \omega^2}(\sin(\omega t) - \sin(\omega_0 t)).$$

We know that this satisfies the detuned equation. Now, taking the limit $\omega \to \omega_0$, we have

$$\begin{aligned}\lim_{\omega\to\omega_0} y_p(t) &= \lim_{\omega\to\omega_0} \frac{\sin(\omega t) - \sin(\omega_0 t)}{\omega_0^2 - \omega^2} \\ &= \lim_{\omega\to\omega_0} \frac{t\cos(\omega t)}{-2\omega} \\ &= -\frac{t}{2\omega_0}\cos(\omega_0 t).\end{aligned}$$

Therefore, a particular integral for the resonant case is

$$y_p(x) = -\frac{x}{2\omega_0}\cos(\omega_0 x).$$

The general rule is that if the forcing term is a linear combination of linearly independent complementary functions, the particular integral is of the form

$$y_p(t) = t \times (\text{non-resonant particular integrals}).$$

Remark. If the homogeneous equation is degenerate, we may need to multiply by higher powers of t to find a particular integral, in the form

$$y_p(t) = t^2 \times (\text{non-resonant particular integrals})$$

for a 2nd order degenerate case.

6.5.2 Equidimensional ODEs

Consider the equidimensional ODE

$$ax^2y'' + bxy' + cy = f(x).$$

We have seen that the complementary functions are of the form

$$y_c(x) = Ax^{k_1} + Bx^{k_2}$$

assuming $k_1 \neq k_2$.

If $f(x) \propto x^m$, we try a particular integral of the form

$$y_p(x) = Cx^m$$

for $m \neq k_1$ and $m \neq k_2$.

For the resonance cases $m = k_1$ or $m = k_2$, then the particular integral is of the form

$$y_p(x) = (\ln x)x^{k_1}$$

which follows from Section 6.4.2.

> ***Remark.*** If the homogeneous equation is degenerate, we may need to multiply by higher powers of $\ln x$ to find a particular integral, in the form
>
> $$y_p(x) = (\ln x)^2 x^{k_1}$$
>
> for a 2nd order degenerate case.

6.6 Variation of Parameters

This is a systematic method to find particular integrals given two linearly independent complementary functions.

Consider

$$y'' + p(x)y' + q(x)y = f(x)$$

with linearly independent complementary functions y_1 and y_2.

We will use solution vectors $\mathbf{Y}_1(x)$ and $\mathbf{Y}_2(x)$ as a basis in phase space at any x to write the solution vector for the particular integral.

We have

$$\mathbf{Y}_p(x) = u(x)\mathbf{Y}_1(x) + v(x)\mathbf{Y}_2(x).$$

The components are

$$\begin{aligned} y_p(x) &= u(x)y_1(x) + v(x)y_2(x) \\ y_p'(x) &= u(x)y_1'(x) + v(x)y_2'(x) \end{aligned}$$

If we differentiate again, we get

$$y_p'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'.$$

Subsituting into the ODE, we have

$$\left(uy_1'' + u'y_1' + vy_2'' + v'y_2'\right) + p\left(uy_1' + vy_2'\right) + q\left(uy_1 + vy_2\right) = f(x).$$

Since y_1, y_2 are complementary functions, we have

$$\begin{aligned} y_1'' + py_1' + qy_1 &= 0 \\ y_2'' + py_2' + qy_2 &= 0. \end{aligned}$$

Therefore,

$$u'y_1' + v'y_2' = f(x).$$

Note that the second component of $\boldsymbol{Y}_p$ must be consistent with the derivative of the first component. Therefore, we have the additional constraint

$$\begin{aligned} u'y_1 + uy_1' + v'y_2 + vy_2' = y_p' &= uy_1' + vy_2' \\ u'y_1 + v'y_2 &= 0. \end{aligned}$$

We now have two equations for u' and v':

$$u'y_1' + v'y_2' = f(x)$$
$$u'y_1 + v'y_2 = 0.$$

We can solve for u' and v':

$$\underbrace{\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}_{\text{fundamental matrix}} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

We can therefore invert the fundamental matrix to get

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$u' = -\frac{y_2 f}{W}$$
$$v' = \frac{y_1 f}{W}.$$

Hence, we can integrate to find u and v:

$$u(x) = -\int^x \frac{y_2(t) f(t)}{W(t)}\, \mathrm{d}t$$
$$v(x) = \int^x \frac{y_1(t) f(t)}{W(t)}\, \mathrm{d}t.$$

The particular integral is therefore

$$y_p(x) = -y_1(x) \int^x \frac{y_2(t) f(t)}{W(t)}\, \mathrm{d}t + y_2(x) \int^x \frac{y_1(t) f(t)}{W(t)}\, \mathrm{d}t.$$

Note that changing the lower limits of the integrals only adds multiples of complementary functions to y_p.

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Example 6.20

Consider

$$y'' + 4y = \underbrace{\sin 2x}_{f(x)}$$

The complementary functions are, as we have seen before

$$y_1 = \sin 2x, \quad y_2 = \cos 2x$$

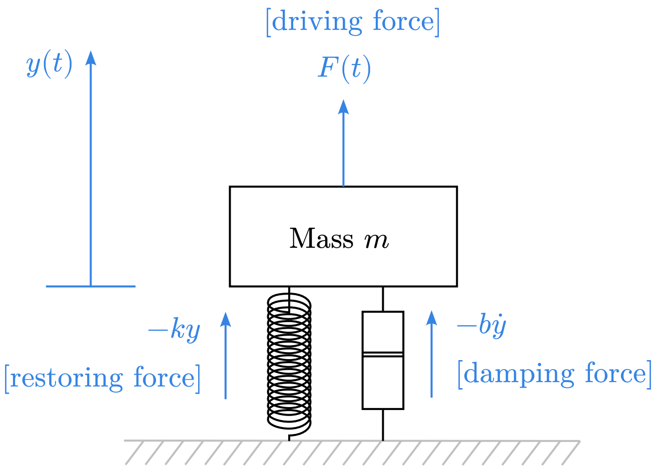
The Wronskian is

$$W(x) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$$
$$= -2\left(\sin^2 2x + \cos^2 2x\right) = -2.$$

Note that the forcing term is resonant.

$$\begin{aligned} y_p &= -\frac{1}{2}\cos 2x \int^x \underbrace{\sin 2u \sin 2u}_{\frac{1}{2}(1-\cos 4u)}\,du - \left(-\frac{1}{2}\right)\sin 2x \int^x \underbrace{\cos 2u \sin 2u}_{\frac{1}{2}\sin 4u}\,du \\ &= -\frac{1}{4}\cos 2x\left[x - \frac{1}{4}\underbrace{\sin 4x}_{2\sin 2x\cos 2x}\right] + \frac{1}{4}\sin 2x\left[-\frac{1}{4}\underbrace{\cos 4x}_{2\cos^2 2x - 1}\right] \\ &= -\frac{1}{4}\underbrace{x\cos 2x}_{x\cdot\text{ C.F.}} + \underbrace{\frac{1}{16}\sin 2x}_{\text{C.F.}}. \end{aligned}$$

6.7 Forced ODEs, Transients and Damping



In the diagram above, by Newton's 2nd Law we have

$$\begin{aligned} m\ddot{y} &= \sum \text{forces} \\ &= -ky - b\dot{y} + F(t). \end{aligned}$$

Rearranging, we get

$$m\ddot{y} + b\dot{y} + ky = F(t).$$

where m, b, k are positive constants.

For $b = 0$ and $F(t) = 0$, we have simple harmonic motion at angular frequency $\omega_0 = \sqrt{\frac{k}{m}}$.

For convenience, we will add in a dimensionless time coordinate $\tau = \omega_0 t$. Then we have $\frac{dy}{dt} = \omega_0 \frac{dy}{d\tau}$. Therefore,

$$y'' + 2\kappa y' + y = f(\tau)$$

where

$$\kappa = \frac{b\omega_0}{2k} = \frac{b}{2m\omega_0}, \quad f(\tau) = \frac{F(\tau)}{k}.$$

6.7.1 Free (Unforced) Response

The behavior is described by one dimensionless parameter κ. We have

$$y'' + 2\kappa y' + y = 0.$$

The characteristic equation is

$$\lambda^2 + 2\kappa\lambda + 1 = 0.$$

The roots are

$$\lambda = -\kappa \pm \sqrt{\kappa^2 - 1}.$$

Depending on the value of κ, we have three cases:

1. **Light damping** (underdamping) when $\kappa < 1$.

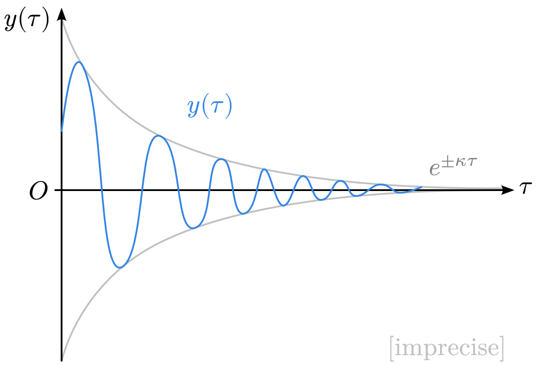
   In this case, the roots are complex, so we can write

   $$\lambda = -\kappa \pm \mathrm{i}\sqrt{1 - k^2}.$$

   So the general solution is

   $$y(\tau) = \mathrm{e}^{-\kappa\tau}\left[A\sin\left(\sqrt{1-\kappa^2}\tau\right) + B\cos\left(\sqrt{1-\kappa^2}\tau\right)\right]$$

   where A, B are constants.



   This is a oscillation at $\omega_{\text{free}} = \sqrt{1-\kappa^2}\omega_0$, which tends to ω_0 as $\kappa \to 0$, with an exponentially decaying amplitude.

   The period is

   $$T = \frac{2\pi}{\omega_{\text{free}}} = \frac{2\pi}{\omega_0\sqrt{1-\kappa^2}}.$$

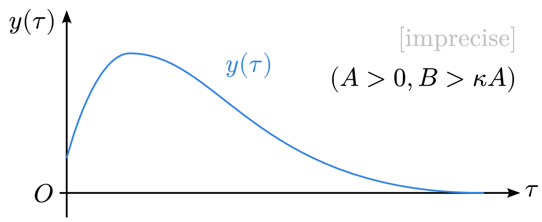
2. **Critical damping** when $\kappa = 1$.

   We have a degenerate root $\lambda = -\kappa$.

   The general solution is

   $$y(\tau) = (A + B\tau)\mathrm{e}^{-\kappa\tau}$$

   where A, B are constants.



3. **Heavy damping** when $\kappa > 1$.

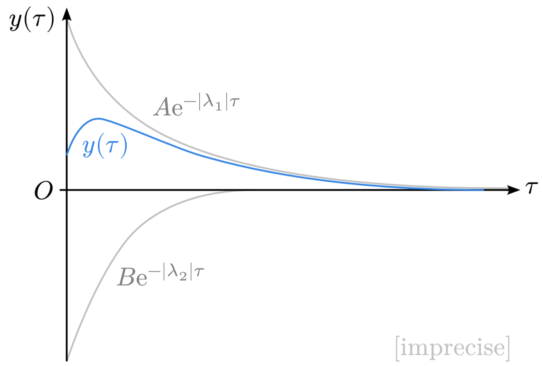
   The roots are real and negative. WLOG take $|\lambda_1| < |\lambda_2|$. Then

$$\lambda_1 = -\kappa + \sqrt{\kappa^2 - 1}, \quad \lambda_2 = -\kappa - \sqrt{\kappa^2 - 1}.$$

   The general solution is

$$y(\tau) = Ae^{-|\lambda_1|\tau} + Be^{-|\lambda_2|\tau}$$

   where A, B are constants. [$-|\lambda|$ indicates that the exponentials decay to zero as $\tau \to \infty$.] Note that $|\lambda_1| < |\lambda_2|$, so the first term dominates the long-term behavior if present.



Unforced response decays eventually in all cases.

6.7.2 Forced Response

Initially, the behavior is determined by C.F. + P.I., which is called the **transient response**.

Over time, C.F. decays, and the behavior is dominated by the P.I., called the **steady-state response**.

Example 6.21

Consider the ODE

$$\ddot{y} + \mu\dot{y} + \omega_0^2 y = \frac{F_0}{m}\sin\omega t.$$

[We can relate this back with $\mu = \frac{b}{m}$ and $\kappa = \frac{\mu}{2\omega_0}$.]

Assume light damping with $\mu < 2\omega_0$. Then the complementary functions are

$$y_c(t) = e^{-\frac{\mu}{2}t}[A\sin(\omega_{\text{free}}t) + B\cos(\omega_{\text{free}}t)]$$

where $\omega_{\text{free}} = \sqrt{\omega_0^2 - \frac{\mu^2}{4}}$.

For the particular integral, we try

$$y_p(t) = \frac{F_0}{m}(C\sin(\omega t) + D\cos(\omega t)).$$

Substituting into the ODE, we get

$$\left[-C\omega^2 - \mu D\omega + \omega_0^2 C\right]\sin(\omega t) + \left[-D\omega^2 + \mu C\omega + \omega_0^2 D\right]\cos(\omega t) = \sin(\omega t).$$

Equating coefficients, we have

$$\begin{aligned} D\left(\omega_0^2 - \omega^2\right) &= \mu C\omega, \\ C\left(\omega_0^2 - \omega^2\right) &= 1 + \mu D\omega. \end{aligned}$$

Eliminating C, we have

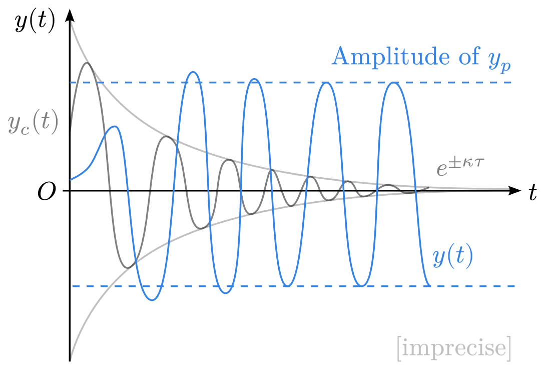
$$\begin{aligned} -\frac{D\left(\omega_0^2 - \omega^2\right)^2}{\mu\omega} &= 1 + \mu D\omega \\ D\left[\mu^2\omega^2 + \left(\omega_0^2 - \omega^2\right)^2\right] &= -\mu\omega \\ D &= -\frac{\mu\omega}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \mu^2\omega^2\right]}. \end{aligned}$$

Therefore,

$$C = \frac{\omega_0^2 - \omega^2}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \mu^2\omega^2\right]}.$$

Hence,

$$y_p(t) = \frac{F_0}{m}\frac{\left[\left(\omega_0^2 - \omega^2\right)\sin(\omega t) - \mu\omega\cos(\omega t)\right]}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \mu^2\omega^2\right]}.$$



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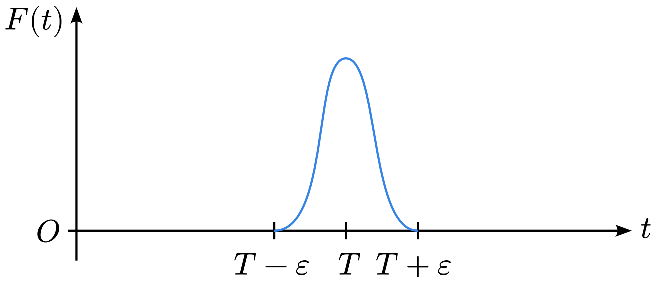
6.8 Impulses and Point Forces

Conside a system that experiences a sudden force between time $t = T - \varepsilon$ and $T + \varepsilon$.



e.g. striking a mass on a spring, or a car going over a curb.

The equation can be in the form

$$m\ddot{y} + b\dot{y} + ky = F(t)$$

which is a forced, damped oscillator.

It is mathamatically convienient to consider the limit of a sudden impluse, as $\varepsilon \to 0$.

We can integrate the ODE from $T - \varepsilon$ to $T + \varepsilon$, and take the limit $\varepsilon \to 0$.

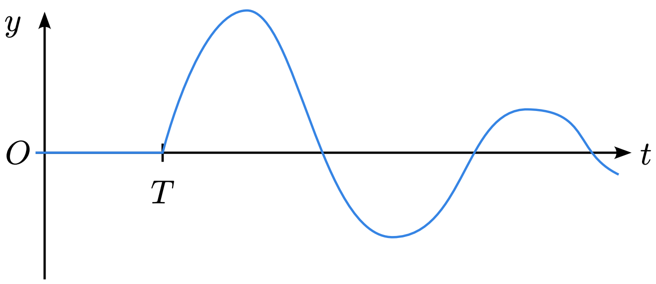
$$\lim_{\varepsilon \to 0+} \left(m[\dot{y}]_{T-\varepsilon}^{T+\varepsilon} + b \underbrace{[y]_{T-\varepsilon}^{T+\varepsilon}}_{\substack{\to 0 \text{ if } y \\ \text{is continuous}}} + k \underbrace{\int_{T-\varepsilon}^{T+\varepsilon} y \, \mathrm{d}t}_{\substack{\to 0 \text{ if } y \\ \text{remains finite}}} \right) = \underbrace{\lim_{\varepsilon \to 0+} \int_{T-\varepsilon}^{T+\varepsilon} F(t) \, \mathrm{d}t}_{\text{impulse}} = I.$$

So we see that

$$\lim_{\varepsilon \to 0^+} \left(m[\dot{y}]_{T-\varepsilon}^{T+\varepsilon} \right) = I,$$

and the velocity $\dot{y}$ is discontinuous.

As $\varepsilon \to 0$ only impluse I matters for subsequent motion.



6.8.1 Dirac Delta Function

We shall formalise the idea of an impulsive force.

Consider a family of functions $D(t; \varepsilon)$ such that

$$\lim_{\varepsilon \to 0} D(t; \varepsilon) = 0 \quad \forall t \neq 0$$

and

$$\int_{-\infty}^{\infty} D(t;\varepsilon)\,\mathrm{d}t = 1. \quad \text{(unit impulse)}$$

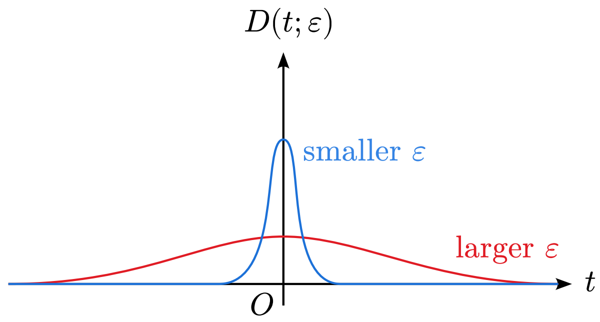
So the impulsive force we considered earlier is:

$$F(t) = ID(t-T;\varepsilon).$$

One example of such a family is

$$D(t;\varepsilon) = \frac{e^{-\frac{t^2}{\varepsilon^2}}}{\varepsilon\sqrt{\pi}}.$$

[See Example Sheet 1, Q14 for normalisation.]



Note that the family of such functions is not unique, but for any family, $\lim_{\varepsilon} \to 0$ yields the Dirac delta function.

Definition 6.22 (Dirac delta function)

The **Dirac delta function** is defined by

$$\delta(t) \to \lim_{\varepsilon\to 0} D(t;\varepsilon).$$

It is technically *not* a function, but a distribution. It only makes sense under an integral.

Proposition 6.23 (Properties of the Dirac delta function)

The Dirac delta function has the following properties:

1. $\delta(t) = 0 \quad \forall t \neq 0.$
2. $\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1.$
3. (Sampling property.) For all functions $g(t)$ that are continuous at $t = 0$,

$$\begin{aligned}\int_{\infty}^{\infty} g(t)\delta(t)\,\mathrm{d}t &= \lim_{\varepsilon\to 0}\int_{\infty}^{\infty} g(t)D(t;\varepsilon)\,\mathrm{d}t \\ &= g(0)\lim_{\varepsilon\to 0}\int_{\infty}^{\infty} D(t;\varepsilon)\,\mathrm{d}t \\ &= g(0).\end{aligned}$$

More generally, for a function $g(t)$ that is continuous at $t = t_0$,

$$\int_a^b g(t)\delta(t-t_0)\,\mathrm{d}t = \begin{cases} g(t_0) & \text{if } a < t_0 < b \\ 0 & \text{if } t_0 < a \text{ or } t_0 > b \\ \text{undefined} & \text{otherwise} \end{cases}$$

6.8.2 Delta Function Forcing

Consider the ODE

$$y'' + p(x)y' + q(x)y = \delta(x) \tag{†}$$

assuming that $p(x)$ and $q(x)$ are continuous.

For $x < 0$ and $x > 0$, we have the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

But there is a discontinuity in y' at $x = 0$:

$$\lim_{\varepsilon\to 0^+}\int_{-\varepsilon}^{\varepsilon}(*)\,\mathrm{d}x = \lim_{\varepsilon\to 0^+}[y']_{-\varepsilon}^{\varepsilon} + p(0)\underbrace{\lim_{\varepsilon\to 0^+}[y]_{-\varepsilon}^{\varepsilon}}_{\substack{\to 0 \text{ if } y \\ \text{is continuous}}} + \underbrace{\lim_{\varepsilon\to 0^+}\int_{-\varepsilon}^{\varepsilon} qy\,\mathrm{d}x}_{\to 0 \text{ if } y \text{ remains finite}} = 1.$$

Therefore,

$$\lim_{\varepsilon\to 0^+}[y']_{-\varepsilon}^{\varepsilon} = 1.$$

which is referred to as a **jump condition**.

Remark. The continuity of y at $x = 0$ is required, since otherwise y' is undefined at $x = 0$, and y'' is even worse-behaved.

The general rule for higher order ODEs is that the highest-order term in ODE *addresses* the delta-function forcing.

Example 6.24

Consider the ODE

$$y'' - y = 3\delta\left(x - \frac{\pi}{2}\right)$$

with $y = 0$ at $x = 0$ and $x = \pi$. We wish to find $y(x)$ for $0 \leqslant x \leqslant \pi$.

Note that we will need to solve the two regions $0 \leqslant x < \frac{\pi}{2}$ and $\frac{\pi}{2} < x \leqslant \pi$ separately, and match them at $x = \frac{\pi}{2}$ using the jump condition.

For $0 \leqslant x < \frac{\pi}{2}$, we have $y'' - y = 0$, which solves to $y = A\sinh x$.

For $\frac{\pi}{2} < x \leqslant \pi$, we have $y'' - y = 0$, which solves to $y = C\sinh(\pi - x)$.

We shall now join the two solutions at $x = \frac{\pi}{2}$.

- y is continuous, so

$$A\sinh\left(\frac{\pi}{2}\right) = C\sinh\left(\frac{\pi}{2}\right)$$
$$A = C.$$

- The jump condition gives

$$\lim_{\varepsilon\to 0^+}[y']_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} = 3$$
$$-A\cosh\left(\frac{\pi}{2}\right) - A\cosh\left(\frac{\pi}{2}\right) = 3 \quad \text{(by using our two solutions on either side)}$$
$$-2A\cosh\left(\frac{\pi}{2}\right) = 3$$
$$A = -\frac{3}{2\cosh\left(\frac{\pi}{2}\right)}.$$

Therefore, the solution is

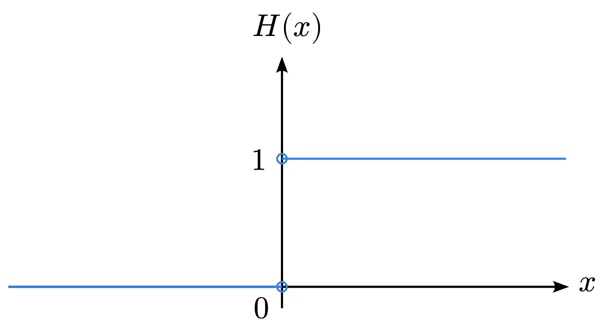
$$y(x) = \begin{cases} -\frac{3\sinh x}{2\cosh(\frac{\pi}{2})} & \text{for } 0 \leqslant x < \frac{\pi}{2} \\ -\frac{3\sinh(\pi - x)}{2\cosh(\frac{\pi}{2})} & \text{for } \frac{\pi}{2} < x \leqslant \pi \end{cases}.$$

6.8.3 Heaviside Step Function $H(x)$

Definition 6.25 (Heaviside step function)

The **Heaviside step function** is defined as

$$H(x) = \int_{-\infty}^{x} \delta(x')\,dx'$$
$$\Rightarrow H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$$



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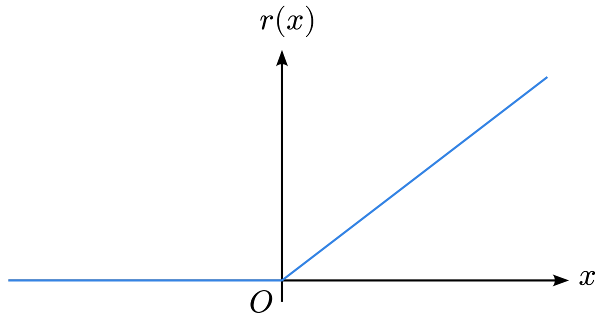
By the fundamental theorem of calculus, we have

$$\frac{dH(x)}{dx} = \delta(x).$$

Definition 6.26 (Ramp function)

The **ramp function** is defined as

$$r(x) = \int_0^x H(x')\,dx' = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geqslant 0. \end{cases}$$



Note hat(t) functions get *smoother* as we integrate them.

6.8.4 Forcing with $H(x)$

Consider the ODE

$$y'' + p(x)y' + q(x)y = H(x)$$

assuming that $p(x)$ and $q(x)$ are continuous at $x = 0$.

So, we have

$$y'' + p(x)y' + q(x)y = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

$$\lim_{\varepsilon\to 0^+} \int_{-\varepsilon}^{\varepsilon} (y'' + p(x)y' + q(x)y)\,dx = \lim_{\varepsilon\to 0^+} \int_{-\varepsilon}^{\varepsilon} H(x)\,dx$$

$$\lim_{\varepsilon\to 0^+} [y']_{-\varepsilon}^{\varepsilon} + p(0) \underbrace{\lim_{\varepsilon\to 0^+} [y]_{-\varepsilon}^{\varepsilon}}_{\substack{\to 0 \text{ if } y \\ \text{is continuous}}} + \underbrace{\lim_{\varepsilon\to 0^+} \int_{-\varepsilon}^{\varepsilon} q(x)y\,dx}_{\to 0 \text{ if } y \text{ remains finite}} = 0$$

$$\lim_{\varepsilon\to 0^+} [y']_{-\varepsilon}^{\varepsilon} = 0.$$

Therefore, y' and y is continuous at $x = 0$. We have $y'' \sim H(x)$ around $x = 0$. So, evaluating both sides around $x = 0$,

$$\lim_{\varepsilon\to 0^+} \left([y'']_{-\varepsilon}^{\varepsilon} + p(0)[y']_{-\varepsilon}^{\varepsilon} + q(0)[y]_{-\varepsilon}^{\varepsilon}\right) = \lim_{\varepsilon\to 0^+} [H(x)]_{-\varepsilon}^{\varepsilon}$$

$$\lim_{\varepsilon\to 0^+} [y'']_{-\varepsilon}^{\varepsilon} = 1.$$

The jump conditions for Heaviside function are

$$\lim_{\varepsilon\to 0^+} [y]_{-\varepsilon}^{\varepsilon} = 0$$

$$\lim_{\varepsilon\to 0^+} [y']_{-\varepsilon}^{\varepsilon} = 0.$$

A typical situation is that $y = 0$ for $x < 0$, and we are asked to find y for $x > 0$. This leads to two arbitrary constants, and we can determine them using the two jump conditions.

Or alternatively, we can solve the ODE for $x > 0$ and $x < 0$ by matching at $x = 0$ using the jump conditions.

7 Higher Order Discrete Equations

Consider a linear, discrete, 2nd order equation with constant coefficients of the form

$$ay_{n+2} + by_{n+1} + cy_n = f_n$$

where a, b, c are constants.

Example 7.1

This may arise when discretising a 2nd order ODE:

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x_n} \approx \frac{y(x_n + h) - 2y(x_n) + y(x_n - h)}{h^2}$$

which we can correspond to

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x_n} \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}.$$

We can solve this using similar methods to solving 2nd order ODEs, with general solution

$$y = y_n^{(c)} + y_n^{(p)}$$

where $y_n^{(c)}$ is the complementary function, and $y_n^{(p)}$ is a particular integral.

7.1 Complementary Functions

The complementary function must satisfy

$$ay_{n+2} + by_{n+1} + cy_n = 0.$$

We can try $y_n^{(c)} \propto k^n$, since k^n is an eigenfunction. This gives the characteristic equation

$$ak^2 + bk + c = 0.$$

The general complementary function is

$$y_n^{(c)} = \begin{cases} Ak_1^n + Bk_2^n & \text{if } k_1 \neq k_2 \\ (A + Bn)k_1^n & \text{if } k_1 = k_2 \end{cases}$$

where k_1, k_2 are the roots of the characteristic equation, and A, B are arbitrary constants.

7.2 Particular integrals

We can find particular integrals based on the form of f_n:

f_n	$y_n^{(p)}$
k^n	$Ak^n \quad (\text{if } k \neq k_1 \text{ or } k_2)$
k_1^n	Ank_1^n
$n^p \quad \left(p \in \mathbb{Z}_{\geqslant 0}\right)$	$An^p + Bn^{p-1} + \ldots + C_n + D$

Example 7.2 (Fibbonacci Sequence)

Consider the sequence with conditions

$$y_n = y_{n-1} + y_{n-2}, \quad y_0 = 1, \quad y_1 = 1.$$

[The sequence starts with $1, 1, 2, 3, 5, 8, 13, \ldots$]

We can rewrite this as

$$y_{n+2} - y_{n+1} - y_n = 0.$$

The characteristic equation is

$$k^2 - k - 1 = 0,$$

with roots

$$k_1 = \frac{1+\sqrt{5}}{2}, \quad k_2 = \frac{1-\sqrt{5}}{2}.$$

Note that the roots are golden ratios:

$$\varphi_1 = \frac{1+\sqrt{5}}{2}, \quad \varphi_2 = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi_1}.$$

Thus the complementary function is

$$y_n^{(c)} = A\varphi_1^n + B\varphi_2^n.$$

To find A and B, we use the initial conditions:

$$y_0 = A + B = 1,$$

$$y_1 = A\varphi_1 + B\varphi_2 = 1.$$

Solving this system gives

$$A = \frac{\varphi_1}{\sqrt{5}}, \quad B = -\frac{\varphi_2}{\sqrt{5}}.$$

Therefore the closed form solution is

$$\begin{aligned} y_n &= \frac{\varphi_1^{n+1} - \varphi_2^{n+1}}{\sqrt{5}} \\ &= \frac{\varphi_1^{n+1} - \left(-\frac{1}{\varphi_1}\right)^{n+1}}{\sqrt{5}}. \end{aligned}$$

Interestingly, an integer sequence can be expressed in terms of irrational numbers.

Since $\varphi_1 > 1$, $y_n \to \frac{(\varphi_1)^{n+1}}{\sqrt{5}}$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \varphi_1 = \frac{1+\sqrt{5}}{2}.$$

8 Series Solutions

When we cannot find simple closed form solutions to ODEs, series solutions may be useful.

Consider

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

The feasibility of finding series solution around $x = x_0$ depends on nature of $p(x)$, $q(x)$, and $r(x)$ around x_0.

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8.1 Classification of Singular Points

Definition 8.1 (Ordinary point and singular point)

The point $x = x_0$ is an **ordinary point** of the ODE

$$p(x)y'' + q(x)y' + r(x)y = 0$$

if $\frac{q(x)}{p(x)}$ and $\frac{r(x)}{p(x)}$ are both analytic at x_0.

[For the purpose of this course, a function is analytic at x_0 if it have a convergent Taylor series about $x = x_0$.]

Otherwise, $x = x_0$ is a **singular point**.

If x_0 is a singular point, but

$$(x - x_0)\frac{q(x)}{p(x)} \quad \text{and} \quad (x - x_0)^2\frac{r(x)}{p(x)}$$

are analytic, then x_0 is a **regular singular point**; otherwise, it is an **irregular singular point**.

Compare this to an equidimensional equation

$$x^2y'' + axy' + by = 0,$$

where $x = 0$ is a regular singular point. We can state that regular singular points are no more singular than $x = 0$ in equidimensional equations.

Example 8.2

Consider the ODE

$$\underbrace{\left(1 - x^2\right)}_{p}y'' \underbrace{-2x}_{q}y' \underbrace{+2}_{r}y = 0.$$

So we have

$$\frac{q}{p} = \frac{-2x}{1-x^2} = \frac{2x}{x-1}(x+1),$$
$$\lim_{x\to\pm1} \frac{q}{p} = \pm\infty.$$

and

$$\frac{r}{p} = \frac{2}{1-x^2} = -\frac{2}{(x-1)(x+1)},$$
$$\lim_{x\to\pm1} \frac{r}{p} = \pm\infty.$$

Therefore, $x = 1$ and $x = -1$ are singular points.

Now, consider $x = 1$. We have

$$(x-1)\frac{q}{p} = (x-1)\frac{-2x}{1-x^2} = -\frac{2x}{x+1},$$

which is analytic at $x = 1$ with value -1. Also,

$$(x-1)^2\frac{r}{p} = (x-1)^2\frac{2}{1-x^2} = \frac{2(x-1)}{x+1},$$

which is analytic at $x = 1$ with value 0.

Hence, $x = 1$ is a regular singular point. Similarly, $x = -1$ is also a regular singular point.

Example 8.3

Consider the ODE

$$\left(1+\sqrt{x}\right)y'' - 2xy' + 2y = 0.$$

So we have

$$\frac{q}{p} = \frac{-2x}{1+\sqrt{x}},$$
$$\lim_{x\to 0+} \frac{q}{p} = 0$$

It may appear that $x = 0$ is an ordinary point. However, the 2nd derivative of $\frac{q}{p}$ is not defined at $x = 0$. Therfore, $x = 0$ is a singular point. Now, consider

$$(x-0)\frac{q}{p} = x\frac{-2x}{1+\sqrt{x}} = \frac{-2x^2}{1+\sqrt{x}}.$$

It does not have a Taylor series about $x = 0$ again. Hence, $x = 0$ is an irregular singular point.

8.2 Method of Frobenius

Theorem 8.4 (Fuch's Theorem)

1. If $x = x_0$ is an ordinary point of the ODE
$$p(x)y'' + q(x)y' + r(x)y = 0,$$
then there are two linearly independent solutions of the form
$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$
which converge for some neighborhood of x_0. *i.e.* there is a Taylor series solution about x_0.
2. If $x = x_0$ is a regular singular point of the ODE, then there is at least one solution of the form
$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma} = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$
where σ can be real or complex, and $a_0 \neq 0$ [so that σ is unique]. The series converges for some neighborhood of x_0.

   This is called a **Frobenius series**.

   Note that there is no guarantee of two linearly independent solutions in this case.

The series solution method may fail completely at irregular singular points.

Example 8.5 (Series solutions about ordinary point)

Suppose we want to find a series solution about $x = 0$ to the ODE
$$\left(1 - x^2\right)y'' - 2xy' + 2y = 0.$$
Since $x = 0$ is an ordinary point, we expect two linearly independent series solutions.

We shall try
$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$
So we have
$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
and
$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$
For convenience, we shall multiply the ODE by x^2 to get

$$x^2\left(1-x^2\right)y'' - 2x^3y' + 2x^2y = 0.$$

Substituting the series into the ODE gives

$$\sum_{n=2}^{\infty} a_n\left[\left(1-x^2\right)n(n-1)\right]x^n - 2\sum_{n=1}^{\infty} a_n x^2 n x^n + 2\sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{\substack{\cancel{n=4}\\ n=2}}^{\infty}(n-2)(n-3)a_{n-2}x^n - 2\sum_{\substack{\cancel{n=3}\\ n=2}}^{\infty}(n-2)a_{n-2}x^n + 2\sum_{n=2}^{\infty} a_{n-2}x^n = 0.$$

Equating coefficients of x^n for $n \geqslant 2$ gives

$$\begin{aligned}
n(n-1)a_n - (n-2)(n-3)a_{n-2} - 2(n-2)a_{n-2} + 2a_{n-2} &= 0\\
n(n-1)a_n - \left(n^2 - 5n + 6 + 2n - 4 - 2\right)a_{n-2} &= 0\\
n(n-1)a_n &= n(n-3)a_{n-2}\\
a_n &= \frac{n-3}{n-1}a_{n-2}.
\end{aligned}$$

Hence we have a recurrence relation for $n \geqslant 2$. Therefore, a_0 and a_1 are arbitrary constants in the general solution.

Consider the odd terms. Note that $n_3 = 0$. Hence, all odd terms are zero. Therefore, one solution is

$$y(x) = a_1 x. \quad \text{(odd function of } x\text{)}$$

Consider the even terms. We have

$$a_n = \frac{n-3}{n-1}a_{n-2} = \frac{(n-3)(n-5)}{(n-1)(n-3)}a_{n-4} = \frac{n-5}{n-1}a_{n-4}.$$

Therefore

$$a_n = -\frac{1}{n-1}a_0.$$

Hence, the other solution is

$$y(x) = a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \dots\right]. \quad \text{(even function of } x\text{)}$$

Note that

$$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} \pm \frac{x^5}{5} + \dots$$

Therefore, we can write the even solution as

$$y(x) = a_0\left[1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right].$$

Hence, we can write the general solution as

$$y(x) = a_0\left[1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + a_1 x$$

which is a closed form solution as well.

Note the behavior near $x = \pm 1$, near the regular singular points.

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Example 8.6 (Series solutions about regular singular point)

Consider the ODE

$$4xy'' + 2\left(1 - x^2\right)y' - xy = 0.$$

We know that $x = 0$ is a regular singular point. For convenience, multiply the ODE by x to get

$$4x^2y'' + 2\left(1 - x^2\right)(xy') - x^2y = 0.$$

We shall try a Frobenius series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\sigma} \quad (a_0 \neq 0).$$

Since the ODE now looks like an equidimensional equation, we expect to be able to extract a sum when we substitute it in. So we have

$$\sum_{n=0}^{\infty} a_n x^{n+\sigma}\left[4(n+\sigma)(n+\sigma-1) + 2\left(1-x^2\right)(n+\sigma) - x^2\right] = 0.$$

- We shall look at the lowest power of x to determine σ. In this case, it is x^σ with $n = 0$.

$$\begin{aligned} a_0[4\sigma(\sigma-1) + 2\sigma] &= 0 \\ 2a_0\sigma(2\sigma - 1) &= 0 \quad \text{(indicial equation)} \end{aligned}$$

  Since $a_0 \neq 0$, we have $\sigma = 0$ or $\sigma = \frac{1}{2}$.
- The next lowerest power is $x^{\sigma+1}$ with $n = 1$.

$$\begin{aligned} a_1[4(\sigma+1)\sigma + 2(\sigma+1)] &= 0 \\ 2a_1(\sigma+1)(2\sigma+1) &= 0. \end{aligned}$$

  Since $\sigma = 0$ or $\frac{1}{2}$, we must have $a_1 = 0$ in both cases.
- For more generality, consider $x^{n+\sigma}$ with $n \geqslant 2$.

$$\begin{aligned} a_n 4(n+\sigma)(n+\sigma-1) + 2a_n(n+\sigma) - 2a_{n-2}(n-2+\sigma) - a_{n-2} &= 0 \\ 2(n+\sigma)(2n+2\sigma-1)a_n &= (2n+2\sigma-3)a_{n-2}. \end{aligned}$$

  Now we should consider the two cases for σ separately.
  - For $\sigma = 0$, we have

$$a_n = \frac{2n-3}{2n(2n-1)} a_{n-2}.$$

Since $a_1 = 0$, all odd terms are zero. Now, for the even terms, we have

$$a_2 = \frac{1}{4 \times 3} a_0, \quad a_4 = \frac{5}{8 \times 7} a_2 = \frac{5}{8 \times 7 \times 4 \times 3} a_0, \ldots$$

Therefore, the even terms give one (Taylor series) solution

$$y_1(x) = a_0 \left[1 + \frac{x^2}{4 \times 3} + \frac{5x^4}{8 \times 7 \times 4 \times 3} + \ldots \right].$$

- For $\sigma = \frac{1}{2}$, we have

$$a_n = \frac{\cancel{2}(n-1)}{\cancel{2}(2n+1)n} a_{n-2}.$$

Therefore, for the even terms,

$$a_2 = \frac{1}{2 \times 5} a_0, \quad a_4 = \frac{3}{4 \times 9} a_2 = \frac{3 \times 1}{4 \times 9 \times 2 \times 5} a_0, \ldots$$

Again, all odd terms are zero. Therefore, the even terms give another solution (relabelling constant to b_0):

$$y_2(x) = b_0 x^{\frac{1}{2}} \left[1 + \frac{x^2}{2 \times 5} + \frac{3x^4}{4 \times 9 \times 2 \times 5} + \ldots \right].$$

Note that this is not a Taylor series but a Frobenius series solution.

Therefore, there are two linearly independent solutions at this regular singular point.

Remark. In Example 8.6 we have found 2 independent solutions, but it is not generally the case.

8.3 Second Solutions

Note that we are guaranteed to get one series solution about a regular singular point, but whether we get a second linearly independent such solution depends on the roots of the indicial equation, σ_1 and σ_2.

1. If $\sigma_1 - \sigma_2$ is not an integer, then we get two linearly independent series solutions, in the form

$$y_1 = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

$$y_2 = (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n.$$

2. If $\sigma_2 - \sigma_1$ is a non-zero integer, then we get one series solution involving the larger root, say σ_2.

$$y_1 = (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The second solution has the form:

$$y_2 = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} b_n (x - x_0)^n + C y_1 \ln(x - x_0),$$

where C may or may not be zero. C is a constant determined in terms of a_0 and b_0, so that we have two arbitrary constants as required.

3. If $\sigma_1 = \sigma_2 = \sigma$, our solutions is similar to case (2), but the logarithmic term is always present.

Example 8.7 (Series solution as in case (2))

Consider the ODE

$$x^2 y'' - xy = 0.$$

We know that $x = 0$ is a regular singular point. We shall try a Frobenius series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\sigma} \quad (a_0 \neq 0).$$

Substituting into the ODE gives

$$\sum_{n=0}^{\infty} \left[a_n (n+\sigma)(n+\sigma-1) x^{n+\sigma} - a_n x^{n+\sigma+1} \right] = 0.$$

- The lowest power is x^σ with $n = 0$.

$$a_0 \sigma(\sigma - 1) = 0$$
$$\sigma(\sigma - 1) = 0 \quad \text{(indicial equation)}$$

  Therefore, $\sigma_1 = 0$ and $\sigma_2 = 1$. Since $\sigma_2 - \sigma_1 = 1$ is a non-zero integer, we are in case (2).

- For $n \geqslant 1$, we get

$$a_n (n+\sigma)(n+\sigma-1) - a_{n-1} = 0$$

  - For $\sigma_2 = 1$, we have

$$a_n = \frac{a_{n-1}}{n(n+1)} = \frac{a_0}{n!(n+1)!}.$$

    Therefore,

$$y_1 = a_0 x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \ldots \right)$$

    which is a Taylor series solution.

  - For $\sigma_1 = 0$, we have

$$(n(n-1)) a_n = a_{n-1}.$$

    Note that when $n = 1$, we have $0 = a_0$, which is a contradiction. Therefore, we cannot find a series solution for $\sigma = 0$. Hence consider the form

$$y_2 = \sum_{n=0}^{\infty} b_n x^n + C y_1 \ln(x).$$

We can determine $\{b_n\}$ and C by direct substitution into the ODE, or we can use the reduction of order method discussed earlier.

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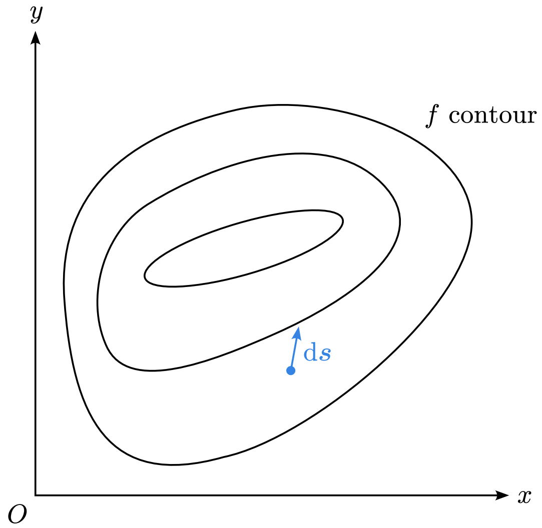
9 Multivariate Functions: Applications

In this section we will discuss

- directional derivatives
- extrema
- coupled systems of 1st order ODEs
- partial differential equations

9.1 Directional derivatives

Consider $f(x, y)$ and a vector displacement $\mathrm{d}\boldsymbol{s} = (\mathrm{d}x, \mathrm{d}y)$.



The infinestimal change in f along $\mathrm{d}\boldsymbol{s}$ is given by

$$\begin{aligned} \mathrm{d}f &= \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y \quad \text{(multivariate chain rule)} \\ &= (\mathrm{d}x, \mathrm{d}y) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \\ &= \mathrm{d}\boldsymbol{s} \cdot \boldsymbol{\nabla} f \end{aligned}$$

where we have defined the gradient operator

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

If we write $\mathrm{d}\boldsymbol{s} = \mathrm{d}s\hat{\boldsymbol{s}}$ where $\hat{\boldsymbol{s}}$ is a unit vector in the direction of $\mathrm{d}\boldsymbol{s}$, then we have

$$\mathrm{d}f = \mathrm{d}s[\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla} f].$$

Definition 9.1 (Directional derivative)

The **directional derivative** of f in the direction of the unit vector $\hat{\boldsymbol{s}}$ is defined as

$$\frac{df}{ds} = \hat{s} \cdot \nabla f = \cos\theta |\nabla f|$$

where θ is the angle between $\hat{s}$ and ∇f.

This is the rate of change of $f(x, y)$ in the direction of $\hat{s}$.

Remark. We can define the gradient vector ∇f geometrically in the other way round, as the vector such that

$$\frac{df}{ds} = \hat{s} \cdot \nabla f \quad \forall \hat{s}.$$

Proposition 9.2 (Properties of gradient vector)

1. The direction of ∇f is the direction of maximum increase of f.
2. The magnitude of ∇f is the maximum rate of change of f, *i.e.*

$$|\nabla f| = \max_{\forall\theta} \frac{df}{ds}$$

3. If $\hat{s}$ is parallel to contours of $f(x, y)$, then

$$\frac{df}{ds} = 0 = \hat{s} \cdot \nabla f.$$

   Therefore, ∇f is perpendicular to the contours of $f(x, y)$.

9.2 Stationary Points

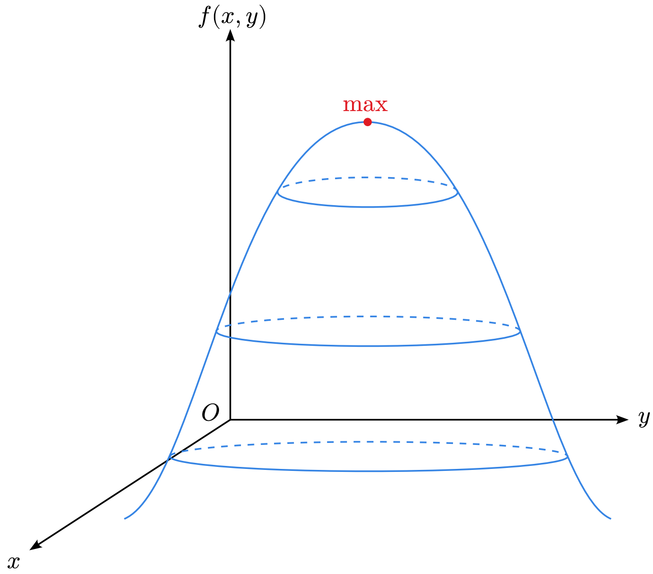
There is always at least one direction where $\frac{df}{ds}$ is zero at a given point, namely the direction parallel to the contours of $f(x, y)$ at that point.

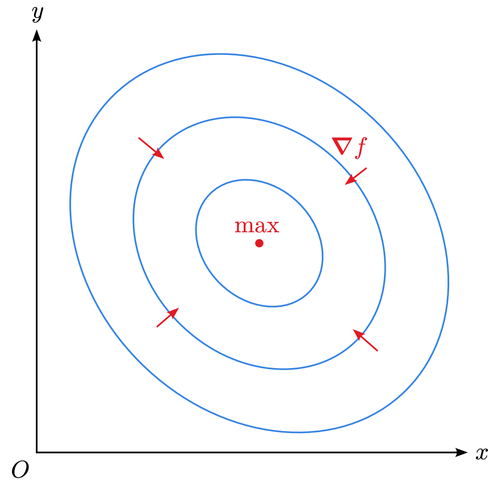
So stationary points are the points where

$$\frac{df}{ds} = 0 \quad \forall \hat{s} \Leftrightarrow \nabla f = \mathbf{0}.$$

9.2.1 Types of Stationary Points

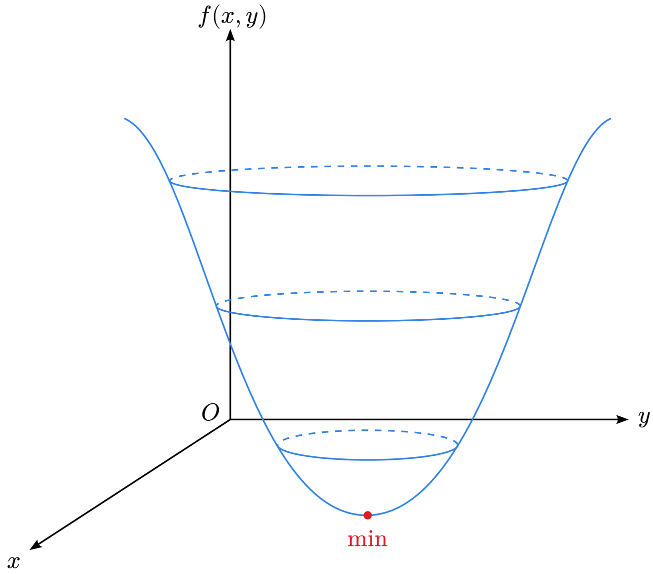
Local Maxima

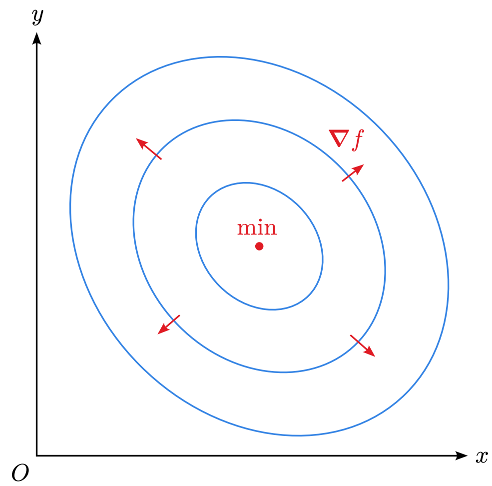




Contours are usually locally elliptical

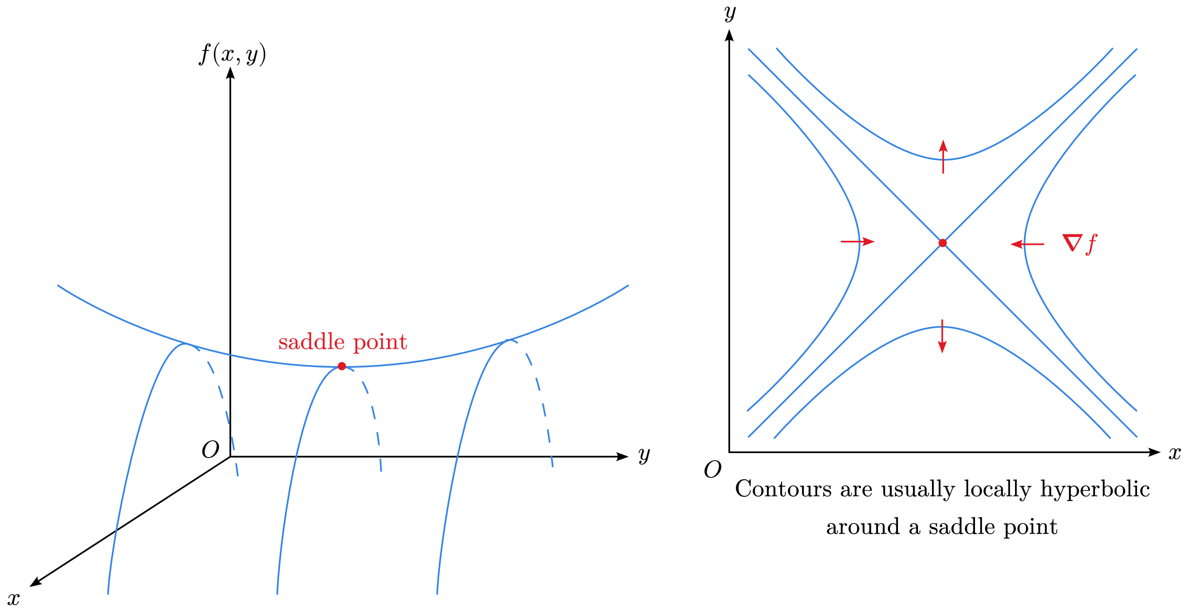
Local Minima





Contours are usually locally elliptical

Saddle Points (not a local extrema)



Contours are usually locally hyperbolic around a saddle point

Note that contours cross at (and only at) saddle points.

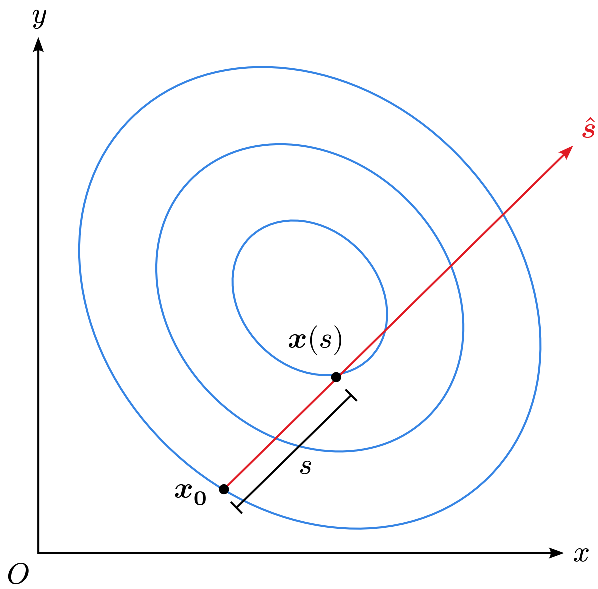
9.3 Classification of Stationary Points

We shall consider how f change in vicinity of a stationary point.

9.3.1 Taylor Series for Multivariate Functions

Consider how $f(x, y)$ varies along the line

$$\mathbf{x}(s) = \mathbf{x}_0 + s\hat{\mathbf{s}}.$$



Along the line, $f(x(s), y(s))$ is a function of s, and we can use the usual Taylor series for single variable functions:

$$f(\mathbf{x}_0 + s\hat{\mathbf{s}}) = f(\mathbf{x}_0) + s\frac{\mathrm{d}f}{\mathrm{d}s}\bigg|_{\mathbf{x}_0} + \frac{s^2}{2!}\frac{\mathrm{d}^2 f}{\mathrm{d}s^2}\bigg|_{\mathbf{x}_0} + \ldots$$

$$= f(\mathbf{x}_0) + \underbrace{s\hat{\mathbf{s}} \cdot \nabla f|_{\mathbf{x}_0}}_{(1)} + \underbrace{\frac{s^2}{2!}(\hat{\mathbf{s}} \cdot \nabla)(\hat{\mathbf{s}} \cdot \nabla f)|_{\mathbf{x}_0}}_{(2)} + \ldots$$

We have

$$\begin{aligned}(1) : s\hat{\mathbf{s}} \cdot \nabla f &= \delta\mathbf{x} \cdot \nabla f \\ &= \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}\end{aligned}$$

$$\text{where } \delta\mathbf{x} = s\hat{\mathbf{s}} = (\delta x, \delta y).$$

and also

$$\begin{aligned}(2) : s^2(\hat{\mathbf{s}} \cdot \nabla)(\hat{\mathbf{s}} \cdot \nabla f) &= s^2\left(\hat{s}_x \frac{\partial}{\partial x} + \hat{s}_y \frac{\partial}{\partial y}\right)\left(\hat{s}_x \frac{\partial f}{\partial x} + \hat{s}_y \frac{\partial f}{\partial y}\right) \\ &= (\delta x)^2 \frac{\partial^2 f}{\partial x^2} + 2(\delta x)(\delta y)\frac{\partial^2 f}{\partial x \partial y} + (\delta y)^2 \frac{\partial^2 f}{\partial y^2} \\ &= (\delta x \;\; \delta y)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}\end{aligned}$$

Definition 9.3 (Hessian Matrix)

The **Hessian matrix** of $f(x, y)$ is defined as

$$\mathbf{H} = \nabla\nabla f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

where $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$, etc.

This is a symmetric matrix since $f_{xy} = f_{yx}$.

Thus, the multivariate Taylor series expansion of $f(x, y)$ about the point $\mathbf{x}_0$ is

$$f(x_0 + \delta x, y_0 + \delta y) = f(x_0, y_0) + \left(\delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}\right)\bigg|_{\mathbf{x}_0} + \left(\frac{1}{2}\right)\left[(\delta x)^2 \frac{\partial^2 f}{\partial x^2} + 2(\delta x)(\delta y)\frac{\partial^2 f}{\partial x \partial y} + (\delta y)^2 \frac{\partial^2 f}{\partial y^2}\right]\bigg|_{\mathbf{x}_0} + \ldots$$

We can also write this in coordinate-independent form as

$$f(\mathbf{x}_0 + \delta\mathbf{x}) = f(\mathbf{x}_0) + \delta\mathbf{x} \cdot \nabla f|_{\mathbf{x}_0} + \left(\frac{1}{2}\right)(\delta\mathbf{x})^{\mathsf{T}}\mathbf{H}|_{\mathbf{x}_0}(\delta\mathbf{x}) + \ldots$$

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9.3.2 Nature of Stationary Points and the Hessian

Suppose $\mathbf{x}_0$ is a stationary point with

$$\nabla f|_{\mathbf{x}_0} = \mathbf{0}.$$

Around $\mathbf{x}_0$:

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \left(\frac{1}{2}\right)(\delta\mathbf{x})^{\mathsf{T}}\mathbf{H}|_{\mathbf{x}_0}(\delta\mathbf{x})$$

where $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$.

Definition 9.4 (Definiteness of a Matrix)

A real symmetric matrix $\mathbf{H}$ is **positive definite** if

$$\mathbf{x}^{\mathsf{T}}\mathbf{H}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}.$$

It is **negative definite** if

$$\mathbf{x}^{\mathsf{T}}\mathbf{H}\mathbf{x} < 0 \quad \forall \mathbf{x} \neq \mathbf{0}.$$

Otherwise, it is **indefinite**.

- If $\mathbf{H}$ is positive definite at $\mathbf{x}_0$, then $f(\mathbf{x}) > f(\mathbf{x}_0)$ for all $\mathbf{x}$ near $\mathbf{x}_0$, so $\mathbf{x}_0$ is a local minimum.
- If $\mathbf{H}$ is negative definite at $\mathbf{x}_0$, then $f(\mathbf{x}) < f(\mathbf{x}_0)$ for all $\mathbf{x}$ near $\mathbf{x}_0$, so $\mathbf{x}_0$ is a local maximum.
- If $\mathbf{H}$ is indefinite at $\mathbf{x}_0$, then it may be a maximum, minimum or saddle point.

9.3.2.1 Definiteness and Eigenvalues

If $\mathbf{H}$ is a real symmetric matrix, then we can diagonalise it by an orthogonal transformation (by results in IA Vectors and Matrices). Using coordinates along the principal axes (eigenvectors), in N dimensions:

$$\begin{aligned} \delta x^{\mathsf{T}}\mathbf{H}\delta x &= \begin{pmatrix} \delta x_1 & \delta x_2 & \dots & \delta x_N \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_N \end{pmatrix} \\ &= \sum_{i=1}^{N} \lambda_i (\delta x_i)^2. \end{aligned}$$

Hence,

- $\mathbf{H}$ is positive definite iff all eigenvalues $\lambda_i > 0$ (minimum),
- $\mathbf{H}$ is negative definite iff all eigenvalues $\lambda_i < 0$ (maximum),
- If all eigenvalues are non-zero, but are of mixed signs, then this corresponds to a saddle point.
- If any of the eigenvalues are zero, then we need higher order terms in the Taylor series to classify the stationary point.

Example 9.5

Consider $f(x, y) = x^2 + y^4$.

This function has a (global) minimum at $(0, 0)$ since $f(x, y) \geqslant 0$ for all (x, y). We have

$$\boldsymbol{\nabla} f = \left(2x, 4y^3\right), \quad \mathbf{H} = \begin{pmatrix} 2 & 0 \\ 0 & 12y^2 \end{pmatrix}.$$

At the stationary point $(0, 0)$, the Hessian matrix is

$$\mathbf{H} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

This has eigenvalues $\lambda_1 = 2 > 0$ and $\lambda_2 = 0$. Therefore, the Hessian is positive semi-definite, and we need to consider higher order terms to classify the stationary point.

9.3.2.2 Definiteness and Signature

An alternative method to determine definiteness without having to compute eigenvalues is to use signatures.

Definition 9.6 (Signature)

The **signature** of **H** is the parttern of signs of the ordered determinants of the leading principal minors of **H**.

Example 9.7

For a function $f(x_1, x_2, ..., x_N)$, the signature if given by the signs of

$$\left|f_{x_1x_1}\right|, \quad \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{vmatrix}, \quad ..., \quad \begin{vmatrix} f_{x_1x_1} & \cdots & f_{x_1x_N} \\ \vdots & \ddots & \vdots \\ f_{x_Nx_1} & \cdots & f_{x_N,x_N} \end{vmatrix}.$$

We shall call these determinants $|\mathbf{H}_1|, |\mathbf{H}_2|, ..., |\mathbf{H}_N| = |\mathbf{H}|$.

Proposition 9.8 (Sylvester's Criterion)

Let **H** be a real symmetric matrix of size $N \times N$. Then

$$\mathbf{H} \text{ is a positive definite} \Leftrightarrow \text{signature is } +, +, +, +, ..., +$$
$$\mathbf{H} \text{ is a negative definite} \Leftrightarrow \text{signature is } -, +, -, +, ..., (-1)^N$$

9.3.3 Contours Near Stationary Points

Suppose $f(x, y)$ has a stationary point at $\mathbf{x}_0 = (x_0, y_0)$. Using coordinates aligned with the principal axes of the Hessian matrix at $\mathbf{x}_0$, we have

$$\mathbf{H}(\mathbf{x}_0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Assume that the eigenvalues are non-zero. Then, consider

$$\mathbf{x} = \mathbf{x}_0 + (\xi, \eta),$$

then around $\mathbf{x}_0$ we have

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \left(\frac{1}{2}\right)\left(\lambda_1 \xi^2 + \lambda_2 \eta^2\right).$$

On contours near $\mathbf{x}_0$, since f is constant, we have

$$\lambda_1 \xi^2 + \lambda_2 \eta^2 = \text{constant}$$

- At a maximum or minimum, λ_1 and λ_2 have the same sign, and the contours are ellipses.
- At a saddle point, λ_1 and λ_2 have opposite signs, and the contours are hyperbolae.

Example 9.9

Consider the stationary points of $f(x, y) = 4x^3 - 12xy + y^2 + 10y + 6$.

We have

$$f_x = 12x^2 - 12y, \quad f_y = -12x + 2y + 10.$$

The stationary points are found by solving $f_x = 0$ and $f_y = 0$ simultaenously:

$$\begin{aligned} f_x &= 0 \Rightarrow y = x^2 \\ f_y &= 0 \Rightarrow -12x + 2y + 10 = 0 \\ &\Rightarrow -12x + 2x^2 + 10 = 0 \\ &\Rightarrow x^2 - 6x + 5 = 0 \\ &\Rightarrow (x-1)(x-5) = 0 \\ &\Rightarrow x = 1, 5 \end{aligned}$$

Thus, the stationary points are at $(1, 1)$ and $(5, 25)$.

We have

$$f_{x,x} = 24x, \quad f_{x,y} = -12, \quad f_{y,y} = 2.$$

Hence, the Hessian matrix is

$$\mathbf{H} = \begin{pmatrix} 24x & -12 \\ -12 & 2 \end{pmatrix}.$$

At the stationary point $(1, 1)$, we have

$$\mathbf{H} = \begin{pmatrix} 24 & -12 \\ -12 & 2 \end{pmatrix}.$$

The leading principal minors are

$$|\mathbf{H}_1| = 24 > 0, \quad |\mathbf{H}| = 24 \times 2 - (-12)^2 = 96 < 0.$$

Thus, the signature is $+, -$, so it is indefinite. See that $|\mathbf{H}| \neq 0$, [so that eigenvalues are all non-zero,] and hence $(1, 1)$ is a saddle point.

At the stationary point $(5, 25)$, we have

$$\mathbf{H} = \begin{pmatrix} 120 & -12 \\ -12 & 2 \end{pmatrix}.$$

The leading principal minors are

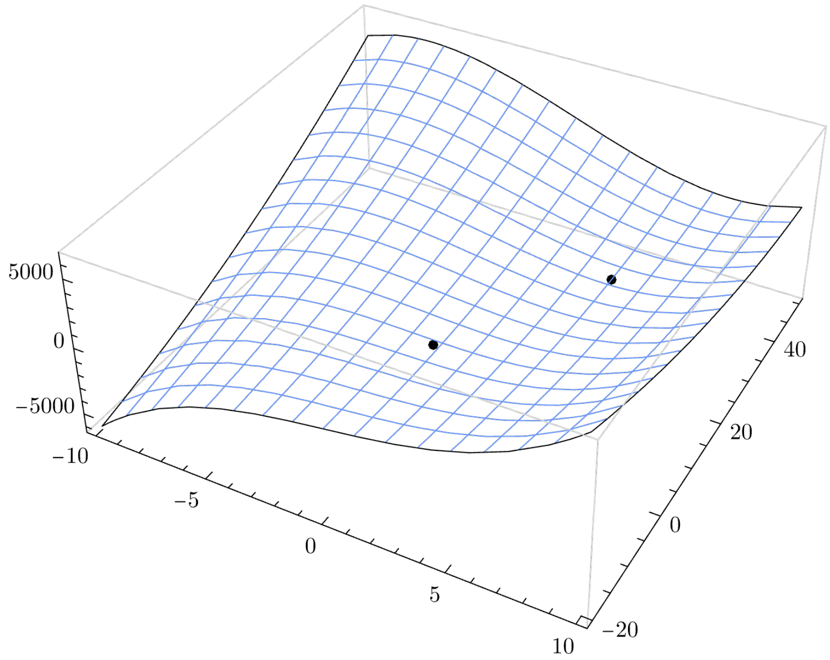
$$|\mathbf{H}_1| = 120 > 0, \quad |\mathbf{H}| = 120 \times 2 - (-12)^2 = 96 > 0.$$

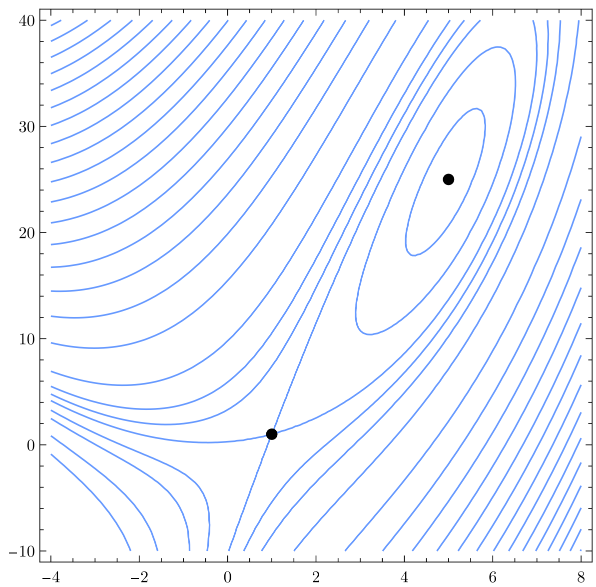
The signature is $+, +$, so it is positive definite, and hence $(5, 25)$ is a local minimum.

Near the saddle points, the contours satisfy

$$24(\delta x)^2 - 24(\delta x)(\delta y) + 2(\delta y)^2 = \text{constant}.$$

Here are some plots of the function and its contours:





9.4 Systems of Linear ODEs

Consider $y_1(t)$ and $y_2(t)$ with

$$\begin{aligned}\dot{y}_1 &= ay_1 + by_2 + f_1(t)\\ \dot{y}_2 &= cy_1 + dy_2 + f_2(t)\end{aligned}$$

where a, b, c, d are constants. We can write this in vector form as

$$\dot{\mathbf{Y}} = \mathbf{M}\mathbf{Y} + \mathbf{F}.$$

where

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

There are two ways to solve this system:

1. Convert to a single higher order ODE for one variable.

   We have

$$\begin{aligned}\ddot{y}_1 &= a\dot{y}_1 + b\dot{y}_2 + \dot{f}_1\\ &= a\dot{y}_1 + b(cy_1 + dy_2 + f_2) + \dot{f}_1\\ &= a\dot{y}_1 + bcy_1 + d(\dot{y}_1 - ay_1 - f_1) + bf_2 + \dot{f}_1\\ \ddot{y}_1 - (a+d)\dot{y}_1 + (ad - bc)y_1 &= \dot{f}_1 - df_1 + bf_2\end{aligned}$$

   Now we have a linear 2nd order ODE with constant coefficients.

2. Solve directly with matrix methods. [This may be more convenient.]

> ***Remark.*** Under some cases, we write higher order ODE as a set of 1st order ODEs, esentially reversing the process above.
>
> **Example 9.10**
>
> Consider the equation
>
> $$\ddot{y} + a\dot{y} + by = f.$$
>
> We can let $y_1 := y$, $y_2 := \dot{y}$ and $\mathbf{Y} = \binom{y_1}{y_2}$. We then have
>
> $$\begin{aligned}\dot{y}_1 &= y_2\\ \dot{y}_2 &= -by_1 - ay_2 + f\end{aligned}$$
>
> Hence,
>
> $$\dot{\mathbf{Y}} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\mathbf{Y} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

9.4.1 Matrix methods

Consider

$$\dot{\boldsymbol{Y}} = \mathbf{M}\boldsymbol{Y} + \boldsymbol{F}(t)$$

where $\mathbf{M}$ is a constant matrix.

1. Write $\boldsymbol{Y} = \boldsymbol{Y}_c + \boldsymbol{Y}_p$ where $\boldsymbol{Y}_c$ is the complementary function satisfying $\dot{\boldsymbol{Y}}_c = \mathbf{M}\boldsymbol{Y}_c$, and $\boldsymbol{Y}_p$ is a particular integral.
2. Look for $\boldsymbol{Y}_c$ of the form $\boldsymbol{Y}_c = \boldsymbol{v}e^{\lambda t}$ where $\boldsymbol{v}$ is a constant vector. Then,

   $$\dot{\boldsymbol{Y}}_c = \lambda \boldsymbol{v} e^{\lambda t} = \lambda \boldsymbol{Y}_c = \mathbf{M}\boldsymbol{Y}_c$$

   Since $\lambda \boldsymbol{Y}_c = \mathbf{M}\boldsymbol{Y}_c$ holds for all t, taking $t = 0$ we have

   $$\mathbf{M}\boldsymbol{v} = \lambda \boldsymbol{v}.$$

   Hence λ is an eigenvalue of $\mathbf{M}$, and $\boldsymbol{v}$ is the corresponding eigenvector.

   For a system of n equations, we have n such complementary functions if eigenvalues are distinct.
3. Find a $\boldsymbol{Y}_p$ that satisfies $\dot{\boldsymbol{Y}}_p = \mathbf{M}\boldsymbol{Y}_p + \boldsymbol{F}(t)$ by trying an appropriate form.

Example 9.11

Consider

$$\dot{\boldsymbol{Y}} = \underbrace{\begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix}}_{\mathbf{M}} \boldsymbol{Y} + \underbrace{\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{t}}_{\boldsymbol{F}}.$$

Write $\boldsymbol{Y} = \boldsymbol{Y}_c + \boldsymbol{Y}_p$, and for $\boldsymbol{Y}_c$ consider $\boldsymbol{Y}_c = \boldsymbol{v}e^{\lambda t}$.

Then,

$$\mathbf{M}\boldsymbol{v} = \lambda \boldsymbol{v} \Rightarrow |\mathbf{M} - \lambda \mathbf{I}| = 0.$$

We have

$$(-4-\lambda)(-2-\lambda) - 24 = \lambda^2 + 6\lambda - 16 = 0$$
$$\lambda_1 = 2, \quad \lambda_2 = -8.$$

The corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -6 \\ 1 \end{pmatrix}.$$

Hence,

$$\boldsymbol{Y}_c = A\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + B\begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8t}.$$

Try $\boldsymbol{Y}_p = \boldsymbol{u}e^{t}$. Then,

$$\begin{aligned}\boldsymbol{u} &= \mathbf{M}\boldsymbol{u} + \begin{pmatrix}4\\1\end{pmatrix}\\ (\mathbf{I}-\mathbf{M})\boldsymbol{u} &= \begin{pmatrix}4\\1\end{pmatrix}\\ \boldsymbol{u} &= (\mathbf{I}-\mathbf{M})^{-1}\begin{pmatrix}4\\1\end{pmatrix}.\end{aligned}$$

Note that an inverse exists since $|\mathbf{I}-\mathbf{M}| \neq 0$ (1 is not an eigenvalue of $\mathbf{M}$). We have f

$$\boldsymbol{u} = \begin{pmatrix}-4\\-1\end{pmatrix}.$$

Thus, the general solution is

$$\boldsymbol{Y} = A\begin{pmatrix}4\\1\end{pmatrix}e^{2t} + B\begin{pmatrix}-6\\1\end{pmatrix}e^{-8t} + \begin{pmatrix}-4\\-1\end{pmatrix}e^{t}.$$

Remark. Note, if $\boldsymbol{F} \propto e^{\lambda t}$ with λ an eigenvalue of $\mathbf{M}$, then we try $\boldsymbol{Y}_p = (\boldsymbol{a} + \boldsymbol{b}t)e^{\lambda t}$ instead.

9.4.2 Non-Degenerate Phase portraits

Definition 9.12 (Phase space)

For n first-order ODEs, the **phase space** is an n-dimensional space with coordinates given by

$$\boldsymbol{Y} = \begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}.$$

Definition 9.13 (Phase portrait)

Phase portraits are solution trajectories in phase space.

For autonomous systems, there is one trajectory through each point in phase space, except at fixed points.

Consider the homogeneous equation

$$\dot{\boldsymbol{Y}} = \mathbf{M}\boldsymbol{Y}.$$

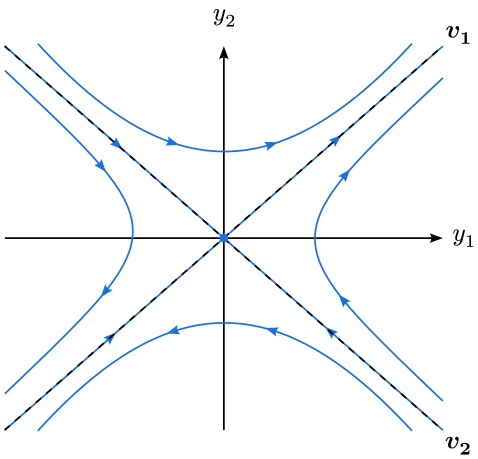
There is a fixed point at $\boldsymbol{Y} = \mathbf{0}$. For $n = 2$, the general solution for $\lambda_1 \neq \lambda_2$ (non-degenerate case) is

$$\boldsymbol{Y}(t) = A\boldsymbol{v}_1 e^{\lambda_1 t} + B\boldsymbol{v}_2 e^{\lambda_2 t}.$$

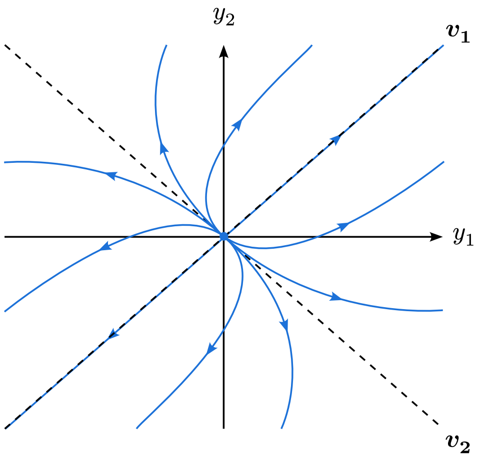
where A, B are constants.

For $\lambda_1 \neq 0, \lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$, we have the following cases:

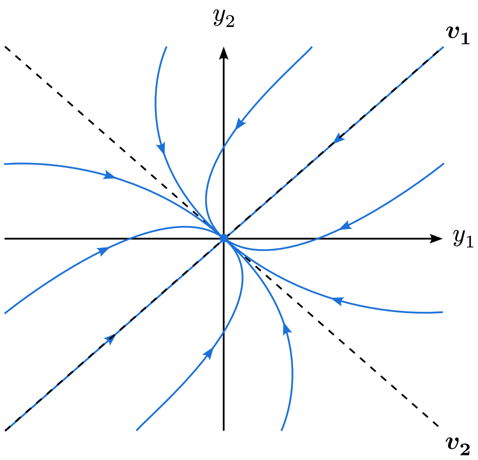
1. λ_1 and λ_2 are real and of opposite signs. WLOG suppose $\lambda_1 > 0 > \lambda_2$. In this case, $\boldsymbol{v}_1, \boldsymbol{v}_2$ can be chosen to be real. The fixed point is a saddle node.



2. λ_2 and λ_2 are real and have the same sign. WLOG suppose $|\lambda_1| > |\lambda_2|$.
   - If both are positive, then the fixed point is an unstable node.



   - If both are negative, then the fixed point is a stable node.

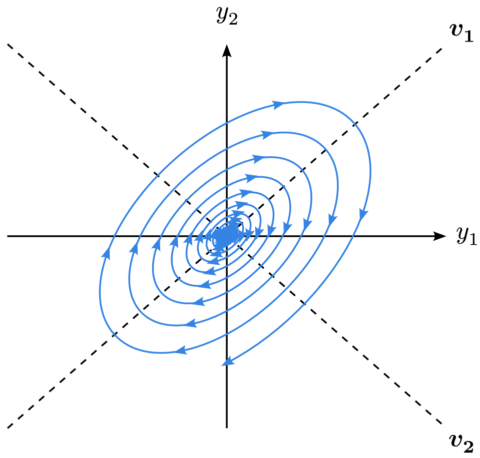


3. λ_1 and λ_2 are complex conjugates, then $\lambda_2 = \overline{\lambda_1}$ and $\mathbf{v_2} = \overline{\mathbf{v_1}}$. Then,

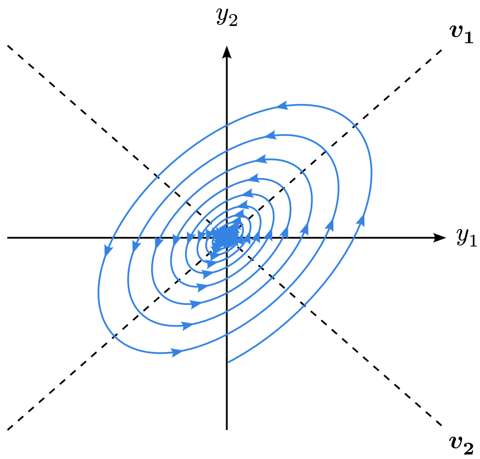
$$\begin{aligned} \mathbf{Y}(t) &= C\mathbf{v_1}e^{\operatorname{Re}(\lambda_1)t}e^{i\operatorname{Im}(\lambda_1)t} + \overline{C}\overline{\mathbf{v_1}}e^{\operatorname{Re}(\lambda_1)t}e^{-i\operatorname{Im}(\lambda_1)t} \\ &= 2e^{\operatorname{Re}(\lambda_1)t}[[c_1\operatorname{Re}(\mathbf{v_1}) - c_2\operatorname{Im}(\mathbf{v_1})]\cos(\operatorname{Im}(\lambda_1)t) - [c_1\operatorname{Im}(\mathbf{v_1}) + c_2\operatorname{Re}(\mathbf{v_1})]\sin(\operatorname{Im}(\lambda_1)t)] \end{aligned}$$

   where $C = c_1 + ic_2$.

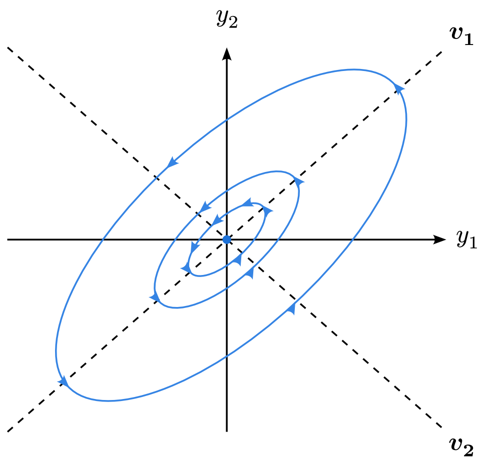
- If $\mathrm{Re}(\lambda_1) > 0$, then we have a unstable spiral.



- If $\mathrm{Re}(\lambda_1) < 0$, then we have a stable spiral.



- If $\mathrm{Re}(\lambda_1) = 0$, then we have a centre, with closed elliptical trajectories.



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In order to determine the direction of motion along the trajectories, we can evaluate $\dot{\mathbf{Y}}$ at some points on the trajectory.

For example, if $\dot{y}_2 > 0$ at $\mathbf{Y} = \binom{1}{0}$, then motion is upwards at that point, so the direction of motion is counter-clockwise.

9.5 Non-Linear Dynamical Systems

We aim to use techniques for linear systems to investigate the nature of equilibrium points.

Consider an autonomous system of 2 non-linear first-order ODEs:

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

where f and g are general non-linear functions of x and y. [They do not depend explicitly on t.]

An equilibrium (fixed) point (x_0, y_0) of the system is a point at which $\dot{x} = 0$ and $\dot{y} = 0$, *i.e.*

$$f(x_0, y_0) = 0 = g(x_0, y_0).$$

We need to solve simultaneously to determine the fixed points.

Stabilities of the fixed points can be deduced from perturbation analysis:

$$(x(t), y(t)) = (x_0 + \xi(t), y_0 + \eta(t))$$

where $\xi(t)$ and $\eta(t)$ are small perturbations. We have

$$\dot{x} = \dot{\xi} = f(x_0 + \xi, y_0 + \eta) \approx \underbrace{f(x_0, y_0)}_{=0 \text{ at fixed point}} + \xi \left.\frac{\partial f}{\partial x}\right|_{x_0, y_0} + \eta \left.\frac{\partial f}{\partial y}\right|_{x_0, y_0}$$
$$\dot{y} = \dot{\eta} = g(x_0 + \xi, y_0 + \eta) \approx \underbrace{g(x_0, y_0)}_{=0 \text{ at fixed point}} + \xi \left.\frac{\partial g}{\partial x}\right|_{x_0, y_0} + \eta \left.\frac{\partial g}{\partial y}\right|_{x_0, y_0}$$

We can write this in matrix form as

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \underbrace{\left.\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\right|_{x_0, y_0}}_{\mathbf{M}} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

This is a linear system of homogenous ODEs, and hence the eigenvalues of **M** determine the stability of the fixed point.

Example 9.14 (Predator-prey model)

Consider a population of prey $x(t)$ and predators $y(t)$ with the equations

$$\text{Prey:} \quad \dot{x} = \underbrace{\alpha}_{\substack{\text{excess births} \\ \text{over natural deaths}}} x - \underbrace{\beta}_{\substack{\text{competition} \\ \text{over scarce} \\ \text{resources}}} x^2 - \underbrace{\gamma xy}_{\substack{\text{deaths due to} \\ \text{predation}}}$$

$$\text{Predators:} \quad \dot{y} = \underbrace{\varepsilon xy}_{\substack{\text{births rate} \\ \text{increases with} \\ \text{predation}}} - \underbrace{\delta}_{\substack{\text{natural} \\ \text{death rate}}} y$$

where $\alpha, \beta, \gamma, \delta, \varepsilon$ are positive constants.

Consider a specific case with

$$\begin{aligned}\dot{x} &= 8x - 2x^2 - 2xy &&= f(x,y)\\ \dot{y} &= xy - y &&= g(x,y)\end{aligned}$$

The fixed points satisfy

$$2x(4-x-y) = 0, \quad y(x-1) = 0.$$

There are three fixed points: $(0,0)$, $(4,0)$ and $(1,3)$.

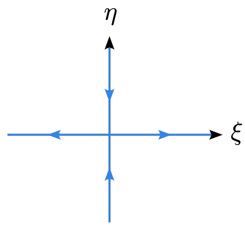
We have

$$\mathbf{M} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 8-4x-2y & -2x \\ y & x-1 \end{pmatrix}.$$

- At $(0,0)$, we have

$$\mathbf{M} = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}.$$

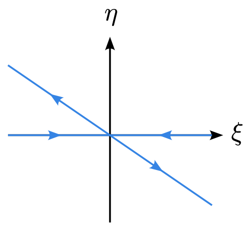
  The eigenvalues are $\lambda_1 = 8 > 0$ and $\lambda_2 = -1 < 0$, with eigenvectors $\mathbf{v}_1 = \binom{1}{0}$ and $\mathbf{v}_2 = \binom{0}{1}$. Thus, it is a saddle point.



- At $(4,0)$, we have

$$\mathbf{M} = \begin{pmatrix} -8 & -8 \\ 0 & 3 \end{pmatrix}.$$

  The eigenvalues are $\lambda_1 = -8 < 0$ and $\lambda_2 = 3 > 0$, with eigenvectors $\mathbf{v}_1 = \binom{1}{0}$ and $\mathbf{v}_2 = \binom{8}{-11}$. Thus, it is a saddle point.



- At $(1,3)$, we have

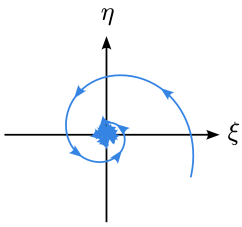
$$\mathbf{M} = \begin{pmatrix} -2 & -2 \\ 3 & 0 \end{pmatrix}.$$

  The eigenvalues are found by solving

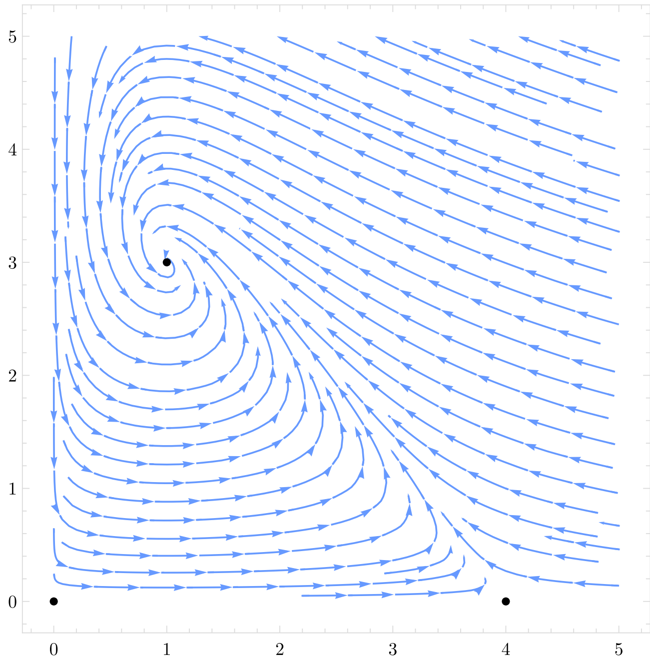
$$|\mathbf{M} - \lambda\mathbf{I}| = \lambda^2 + 2\lambda + 6 = 0$$
$$\lambda = -1 \pm \mathrm{i}\sqrt{5}.$$

Since $\mathrm{Re}(\lambda) = -1 < 0$, the fixed point is a stable spiral.

At $(\xi, \eta) = (1, 0)$, we have $\left(\dot{\xi}, \dot{\eta}\right) = (-2, 3)$, so the motion is counter-clockwise.



Now, we can sketch the overall phase portrait.



9.6 Partial Differential Equations

Partial differential equations (PDEs) involve several independent variables. We will illustrate some ideas with wave equations.

9.6.1 First-Order Wave Equation

Consider $\psi(x, t)$, where x is the spatial coordinate and t is time, with

$$\frac{\partial \psi}{\partial t} - c\frac{\partial \psi}{\partial x} = 0 \tag{‡}$$

where c is a constant with dimensions of velocity.

We can solve this by the method of characteristics. We consider how ψ vary along a path $x(t)$, so that we consider $\psi(x(t), t)$. We have

$$\begin{aligned}\frac{d\psi}{dt} &= \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x}\frac{dx}{dt} && \text{(multivariate chain rule)} \\ &= \frac{\partial\psi}{\partial x}\left(c + \frac{dx}{dt}\right) && \text{(using (*))}\end{aligned}$$

If we choose $x(t)$ such that $\frac{dx}{dt} = -c$, then $x(t) = x_0 - ct$ where x_0 is a constant, and we have

$$\frac{d\psi}{dt} = 0 \Rightarrow \psi(x(t), t) = \psi(x_0, 0) = \text{constant along path.}$$

Paths $x(t)$ where $x(t) = x_0 - ct$ are called characteristics of $(*)$.

Since ψ is constant along characteristics, the general solution of $(*)$ is

$$\psi(x, y) = f(x_0) = f(x + ct)$$

where f is an arbitrary function.

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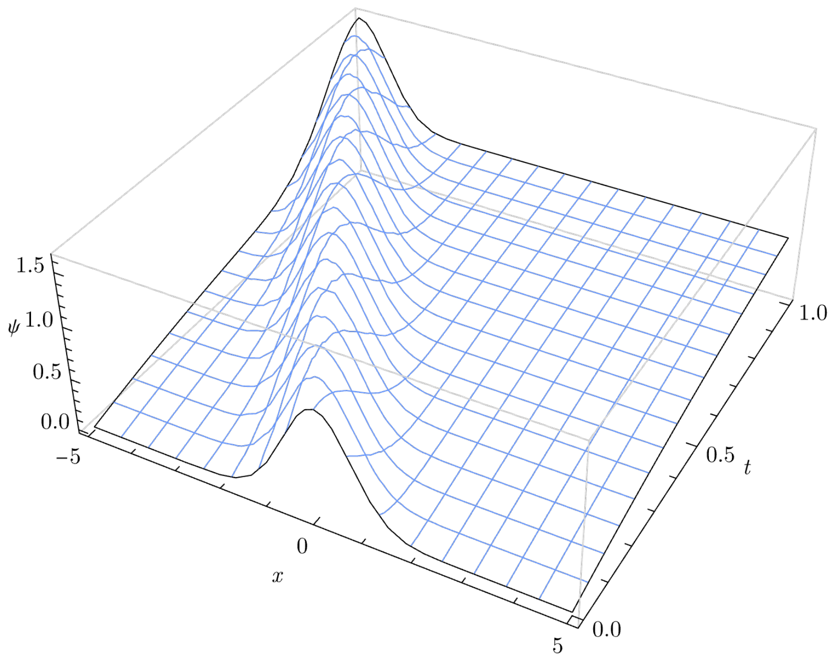
This expression translates the x-dependence of ψ at $t = 0$ to the left by ct at time t.



The solutions are left-moving wave solutions.

Example 9.15 (Unforced wave equation)

We have

$$\frac{\partial\psi}{\partial t} - c\frac{\partial\psi}{\partial x} = 0$$

with $\psi(x, 0) = x^2 - 3$.

The general solution is

$$\psi(x, t) = f(x + ct).$$

Using the initial condition, we have

$$\psi(x,0) = f(x) = x^2 - 3.$$

Therefore, the specific solution is

$$\psi(x,t) = (x+ct)^2 - 3.$$

Example 9.16 (Forced wave equation)

Consider

$$\frac{\partial \psi}{\partial t} + 5\frac{\partial \psi}{\partial x} = \mathrm{e}^{-t}$$

with $\psi(x,0) = \mathrm{e}^{-x^2}$.

The characteristics are of the form $x(t) = x_0 + 5t$.

Along these characteristics, we have

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}t} &= \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial t}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \mathrm{e}^{-t} - 5\frac{\partial \psi}{\partial x} + 5\frac{\partial \psi}{\partial x} \\ &= \mathrm{e}^{-t}.\end{aligned}$$

So this gives

$$\psi = f(x_0) - \mathrm{e}^{-t}$$

where $f(x_0)$ is an arbitrary function.

Using the initial condition at $t = 0$, we have

$$\psi(x,0) = f(x) - 1 = \mathrm{e}^{-x^2} \Rightarrow f(x) = \mathrm{e}^{-x^2} + 1.$$

Thus, the specific solution is

$$\psi(x,t) = \mathrm{e}^{-(x-5t)^2} + 1 - \mathrm{e}^{-t}.$$

9.6.2 Second-Order Wave Equation

A lot of physical systems allow waves to propagate in both directions. This is modelled by second-order wave equations.

Consider

$$\frac{\partial^2 \psi}{\partial t^2} - c^2\frac{\partial^2 \psi}{\partial x^2} = 0$$

where c is a constant with dimensions of velocity.

Since the differential operator can be factorised as

$$\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right),$$

we can write the wave equation as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\psi = 0.$$

These two operators commute, so both $f(x+ct)$ and $g(x-ct)$ are solutions, where f and g are arbitrary functions.

This suggests that the general solution is

$$\psi(x,t) = f(x+ct) + g(x-ct)$$

where f and g are arbitrary functions.

Remark. We can show that this is indeed the most general solution. Let $\xi = x + ct$ and $\eta = x - ct$.

$$\left.\frac{\partial}{\partial x}\right|_t = \underbrace{\left.\frac{\partial \xi}{\partial x}\right|_t}_{1}\left.\frac{\partial}{\partial \xi}\right|_\eta + \underbrace{\left.\frac{\partial \eta}{\partial x}\right|_t}_{1}\left.\frac{\partial}{\partial \eta}\right|_\xi$$

$$\left.\frac{\partial}{\partial t}\right|_x = \underbrace{\left.\frac{\partial \xi}{\partial t}\right|_x}_{c}\left.\frac{\partial}{\partial \xi}\right|_\eta + \underbrace{\left.\frac{\partial \eta}{\partial t}\right|_x}_{-c}\left.\frac{\partial}{\partial \eta}\right|_\xi$$

so, we have

$$\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} = -2c\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} = 2c\frac{\partial}{\partial \xi}.$$

So the wave equation becomes

$$-4c^2\frac{\partial^2 \psi}{\partial \xi \partial \eta} = 0$$

Therefore,

$$\psi(\xi,\eta) = f(\xi) + g(\eta)$$

for arbitrary functions f and g.

Example 9.17

Consider

$$\frac{\partial^2 \psi}{\partial t^2} - c^2\frac{\partial^2 \psi}{\partial x^2} = 0$$

with $\psi(x,0) = \frac{1}{1+x^2}$ and $\frac{\partial \psi}{\partial t}(x,0) = 0$. The general solution is

$$\psi(x,t) = f(x+ct) + g(x-ct).$$

Using the initial conditions, we have

$$\psi(x,0) = f(x) + g(x) = \frac{1}{1+x^2}$$

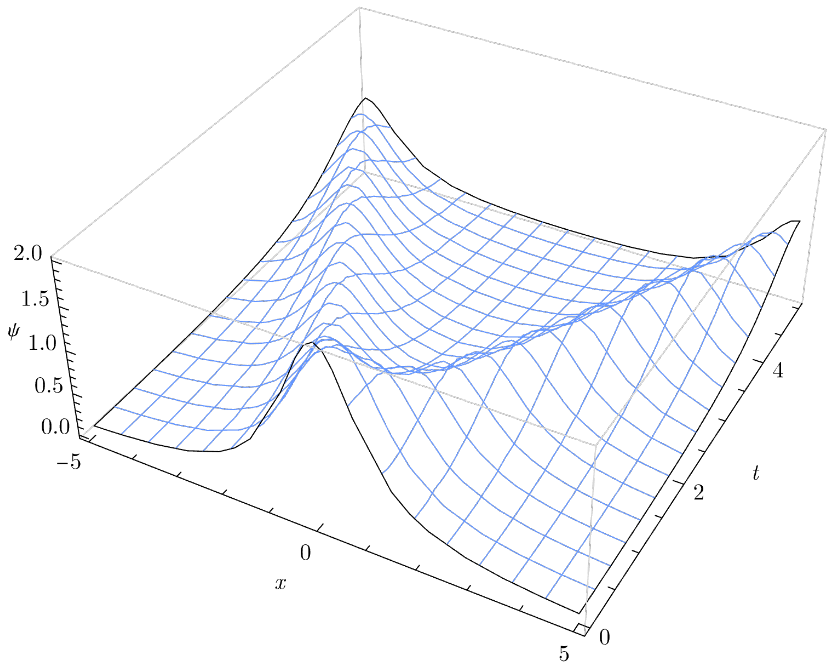
$$\frac{\partial\psi}{\partial t}(x,0) = cf'(x) - cg'(x) = 0 \Rightarrow f(x) - g(x) = A$$

where A is a constant. Solving these two equations gives

$$f(x) = \frac{1}{2(1+x^2)} + \frac{A}{2}, \quad g(x) = \frac{1}{2(1+x^2)} - \frac{A}{2}.$$

Thus, the specific solution is

$$\psi(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right]$$



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