Part IA
Analysis I

Dr Rita Teixeira da Costa

Lent 2026

Version 20260522

These are Zixuan's notes for **Part IA – Analysis I** at the University of Cambridge in 2026. The notes are not endorsed by the lecturers or the University, and all errors are my own.

The latest version of this document is available at academic.micfong.space. Please direct any comments to my CRSid email or use the contact details listed on the site.

This document is typeset using Typst. All figures are created using Inkscape.

Contents

Syllabus and Overview

Lent Term, 2026 [24 Lectures]

Faded topics are not examinable.

Limits and Convergence [6 Lectures]

Sequences and series in ℝ and ℂ. Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test.

Continuity [3 Lectures]

Continuity of real- and complex-valued functions defined on subsets of ℝ and ℂ. The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds.

Differentiability [5 Lectures]

Differentiability of functions from ℝ to ℝ. Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle’s theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor’s theorem from ℝ to ℝ; Lagrange’s form of the remainder. Complex differentiation.

Power Series [4 Lectures]

Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. Direct proof of the differentiability of a power series within its circle of convergence.

Integration [6 Lectures]

Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor’s theorem. Improper integrals.

Lecture 1 · 2026-01-22

1 Numerical Sequences and Series

1.1 Basics

Definition 1.1 (Sequence)

A sequence $(x_n)_{n \in \mathbb{N}}$ on a set X is an enumerated list where each element is in the set X.

In this section, $X \subseteq \mathbb{R}$ or $X \subseteq \mathbb{C}$. A concrete important case is $X = \mathbb{R}$ [real sequences].

We shall now consider the issues of

- convergence, where (x_n) converges to $x \in X$, and
- divergence of sequences.

On convergence:

- we need $|x_n - x|$ to be smaller than any given threshold $\varepsilon > 0$ that we choose;
- for the comparison, only the tail of the sequence matters, i.e. large n behaviour. We can always ignore the first N terms of the sequence, for some N depending on ε.

On divergence to infinity:

- we need $|x_n|$ to clear any threshold $L > 0$ that we choose;
- again, only the tail of the sequence matters. We can always ignore the first N terms of the sequence, for some N depending on L.

Definition 1.2 (Convergence)

We say that (x_n) **converges** to some finite x if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N : |x_n - x| < \varepsilon.$$

We write $x_n \to x$ or $\lim_{n \to \infty} x_n = x$.

Definition 1.3 (Divergence to Infinity)

We say that a real sequence (x_n) **diverges to (positive) infinity** if

$$\forall L > 0, \exists N \in \mathbb{N}, \forall n \geqslant N : x_n > L.$$

We write $x_n \to \infty$.

Remark. We can replace $<$ by $\leqslant$, and replace ε by 2ε, *etc.* in the definitions above without changing their meanings.

Remark. For complex sequences, we can use analogus definitions of diverging to infinity.

Example 1.4

- Consider $x_n = \frac{1}{n}$. Then $x_n \to 0$ because $\forall \varepsilon > 0$,
$$|x_n - 0| = \frac{1}{n} < \varepsilon \quad \forall n \geqslant 1 + \left\lceil \frac{1}{\varepsilon} \right\rceil.$$
- Consider $x_n = \frac{1}{2^n}$. Then $x_n \to 0$ because $\forall \varepsilon > 0$,
$$|x_n - 0| = \frac{1}{2^n} < \varepsilon \quad \forall n \geqslant \max\left\{1, 1 + \left\lceil \log_2\left(\frac{1}{\varepsilon}\right) \right\rceil\right\}.$$
- Consider $x_n = \mathrm{i}\, n$. Then (x_n) diverges to infinity since $\forall L > 0$,
$$\frac{x_n}{\mathrm{i}} = n > L \quad \forall n \geqslant 1 + \lceil L \rceil.$$
- Consider $x_n = (-1)^n$. Then (x_n) does not diverge to ∞, but it does not converge either.

Lemma 1.5

If a sequence (x_n) converges, then the limit is unique.

Proof. Suppose $x_n \to a$ and $x_n \to b$. Take $\varepsilon > 0$, then we have
$$\exists N_1 = N_1(\varepsilon), \forall n \geqslant N_1 : |x_n - a| < \varepsilon,$$
$$\exists N_2 = N_2(\varepsilon), \forall n \geqslant N_2 : |x_n - b| < \varepsilon.$$

In particular, for $n \geqslant \max\{N_1, N_2\}$, both inequalities hold. Then
$$|a - b| = |a - x_n + x_n - b| \leqslant |a - x_n| + |x_n - b| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since ε is arbitrary, we can conclude that $|a - b| = 0$. Hence $a = b$.

Proposition 1.6 (Sandwich Theorems)

Let (x_n), (y_n), (z_n) be real sequences. Then

- If $x_n \leqslant y$ for all n and $x_n \to x$, then $x \leqslant y$.
- If $x_n \to x$, $z_n \to x$, and $x_n \leqslant y_n \leqslant z_n$, then $y_n \to x$.

> ***Important.*** The best conclusion in the case that $x_n < y$ for all n and $x_n \to x$, is still $x \leqslant y$.

Lemma 1.7

Let (x_n) be a complex sequence. Then $x_n \to x$ if and only if $\mathrm{Re}(x_n) \to \mathrm{Re}(x)$ and $\mathrm{Im}(x_n) \to \mathrm{Im}(x)$.

Proof. Recall that for $z \in \mathbb{C}$, $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$.

[$\Rightarrow$] We have that $|\mathrm{Re}\, z| \leqslant |z|$ and $|\mathrm{Im}\, z| \leqslant |z|$. By the defintiion of convergence, the result follows.

[$\Leftarrow$] Note the inequality $|z| \leqslant |\operatorname{Re} z| + |\operatorname{Im} z|$. Fix $\varepsilon > 0$. By definiition of convergence,

$$\exists N_1 = N_1(\varepsilon), \forall n \geqslant N_1 : \left|\operatorname{Re} x_n - \operatorname{Re} x\right| < \varepsilon,$$
$$\exists N_1 = N_2(\varepsilon), \forall n \geqslant N_2 : \left|\operatorname{Im} x_n - \operatorname{Im} x\right| < \varepsilon.$$

Hence $\left|x_n - x\right| \leqslant 2\varepsilon$ for all $n \geqslant \max\{N_1, N_2\}$.

Lemma 1.8

Let $x_n \to x$, $y_n \to y$. Then $x_n + y_n \to x + y$ and $x_n y_n \to xy$. If $x_n \neq 0$ for every n, then $\frac{1}{x_n} \to \frac{1}{x}$.

Proof. The first and third part is left as an exercise.

We shall prove that $x_n y_n \to xy$. We have

$$\begin{aligned}\left|x_n y_n - xy\right| &= \left|x_n y_n - x y_n + x y_n - xy\right| \\ &\leqslant |x|\left|y_n - y\right| + \left|y_n\right|\left|x - x_n\right|.\end{aligned}$$

We have

$$\forall \varepsilon > 0, \exists N_1 = N_1(\varepsilon), \exists N_2 = N_2(\varepsilon), \begin{cases}\forall n \geqslant N_1 : \left|x_n - x\right| < \varepsilon \\ \forall n \geqslant N_2 : \left|y_n - y\right| < \varepsilon.\end{cases}$$

Hence

$$\left|x_n y_n - xy\right| \leqslant \varepsilon(|x| + \left|y_n\right|) \quad \forall n \geqslant \max\{N_1, N_2\}.$$

Lemma 1.9

If $x_n \to x$, then (x_n) must be bounded. *i.e.*

$$\exists M, \forall n : \left|x_n\right| \leqslant M$$

Lecture 2 · 2026-01-24

Proof. Take $\varepsilon = 1$. Then there exists N such that $\forall n \geqslant N$, $\left|x_n - x\right| < 1$. Hence for $n \geqslant N$, we have

$$\left|x_n\right| \leqslant \left|x_n - x\right| + |x| < 1 + |x|.$$

Let

$$M = \max\{\left|x_1\right|, \left|x_2\right|, \ldots, \left|x_{N-1}\right|, 1 + |x|\}.$$

Then $\forall n$, $\left|x_n\right| \leqslant M$.

Hence we can replace $\left|y_n\right|$ with M and the result follows.

Definition 1.10 (Bounded Sequence)

We say (x_n) is **bounded** if $\exists M > 0$ such that $\left|x_n\right| \leqslant M$ for all n. Equivalently, $\sup_{n \geqslant 0}\left|x_n\right| \leqslant M$.

Definition 1.11 (Monotonic Sequence)

We say a real sequence (x_n) is monotone if either

- it is increasing, $x_n \leqslant x_{n+1}$ for every n,
- it is decreasing, $x_n \geqslant x_{n+1}$ for every n.

Proposition 1.12 (Monotone Convergence Theorem)

Every bounded monotone real sequence converges.

Proof. WLOG suppose (x_n) is strictly increasing and bounded above. We will use the supremum axiom, that every non-empty set of $\mathbb{R}$ bounded above has a supremum in $\mathbb{R}$.

Refer to IA Numbers and Sets for a full proof of this proposition.

If we drop the monotonicity condition, we may not have convergence. For example, $x_n = (-1)^n$ is bounded but does not converge. However, we can still extract convergent subsequences from bounded sequences, *e.g.* by taking all even terms in $x_n = (-1)^n$.

1.2 Bolzano-Weierstrass Theorem

Theorem 1.13 (Bolzano-Weierstrass Theorem)

If (x_n) is a real and bounded sequence, then there exists a convergent subsequence.

Definition 1.14 (Subsequence)

A **subsequence** of a sequence (x_n) is a sequence of the form $\left(x_{n_k}\right)$ where (n_k) is a strictly increasing sequence of natural numbers.

Lemma 1.15

If $x_n \to x$, then any subsequence $\left(x_{n_k}\right)$ must converge to the same limit.

Proof. Since $n_k < n_{k+1} \Rightarrow n_{k+1} \geqslant n_k + 1$, by induction, we can show that $n_k \geqslant k$ for all k.

Take $\varepsilon > 0$, then $\exists N = N(\varepsilon)$ such that $\forall n \geqslant N$, $|x_n - x| < \varepsilon$. So if $k \geqslant N$ then $n_k \geqslant k \geqslant N$ and hence $\left|x_{n_k} - x\right| < \varepsilon$.

Hence $\lim_{k\to\infty} x_{n_k} = x$.

Proposition 1.16 (Nested Interval Property)

Take a sequence of nested closed intervals in $\mathbb{R}$: $\forall n$, $I_n \supseteq I_{n+1}$ where $I_n = [a_n, b_n]$.

If $b_n - a_n = |I_n| \to 0$ as $n \to \infty$, then $\bigcap_{n\in\mathbb{N}} I_n$ contains exactly one point.

Proof. This is an application of Monotone Convergence Theorem 1.12.

Since $I_n \supseteq I_{n+1}$ and $I_1 \supseteq I_n$, we have

- $a_n \leqslant a_{n+1}$
- $b_n \geqslant b_{n+1}$
- $a_1 \leqslant a_n \leqslant b_n \leqslant b_1$

Hence,

- (a_n) is increasing and bounded above by b_1,
- (b_n) is decreasing and bounded below by a_1.

So (a_n) and (b_n) converge. Let $a = \lim_{n\to\infty} a_n \in \mathbb{R}$ and $b = \lim_{n\to\infty} b_n \in \mathbb{R}$. Since limits preserve inequalities, we have $a \leqslant b$. Now we shall prove this proposition by considering two aspects.

Existence. For all $k \geqslant n$, $a_k \in [a_k, b_k] \subseteq [a_n, b_n]$. So $a_n \leqslant a_k \leqslant b_n$. Now, as $k \to \infty$, we have $a_n \leqslant a \leqslant b_n$. Since this is true for all n, we have $a \in \bigcap_{n\in\mathbb{N}} I_n$.

Uniqueness. $b_n - a_n \to b - a$ by construction. Since $b_n - a_n \to 0$, we must have $b - a = 0$ because limits are unique. Hence $a = b$.

Thus $x \in \cap_n I_n \Leftrightarrow x \in I_n \ \forall n \Leftrightarrow a_n \leqslant x \leqslant b_n \ \forall n \Rightarrow a \leqslant x \leqslant b = a \Rightarrow x = a = b$.

Proof. [of Bolzano-Weierstrass Theorem 1.13] We are given (x_n) and $M > 0$ such that $|x_n| \leqslant M$ for all n. We will construt a sequence of nested intervals from which we can sample our subsequence, since that will ensure that our subsequence will converge to the unique intersection point of nested intervals.

Let $a_1 = -M, b_1 = M$. Then $I_1 = [-M, M] \supseteq \{x_n : n \in \mathbb{N}\}$.

Now take $c = \frac{a_1+b_1}{2}$. Then at least one of the intervals $[a_1, c]$ and $[c, b_1]$ must contain infinitely many terms of the sequence (x_n). [If both intervals contained only finitely many terms, then the whole interval $[-M, M]$ would contain only finitely many terms, contradicting the fact that (x_n) is an infinite sequence.] Take I_2 to be a half interval that contains infinitely many terms. Continuing inductively gives a sequence of nested intervals $I_n = [a_n, b_n]$ with $b_n - a_n = \frac{M}{2^{n-1}} \to 0$ as $n \to \infty$, and each I_n contains infinitely many terms of the sequence.

By Nested Interval Property 1.16, $\exists! x \in \bigcap_n I_n$. We can now choose $\left(x_{n_k}\right)$ as follows: pick n_1 such that $x_{n_1} \in I_1$, then I_2 has infinitely many elements of (x_n) with indices greater than n_1, so pick $n_2 > n_1$ such that $x_{n_2} \in I_2$. Continuing in this manner gives a subsequence $\left(x_{n_k}\right)$ with $x_{n_k} \in I_k$ for all k.

By construction, $x_{n_k} \in I_k$ for every k, so $x_{n_k} \in \bigcap_{n\leqslant k} I_n$, so $x_{n_k} \to x$ as $k \to \infty$.

Lecture 3 · 2026-01-27

Remark. The Bolzano-Weierstrass Theorem also works for complex sequences.

1.3 Cauchy Sequences

Definition 1.17 (Cauchy Sequence)

A sequence $(x_n) \in \mathbb{C}$ is **Cauchy** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geqslant N : |x_n - x_m| < \varepsilon.$$

Example 1.18

- $x_n = \frac{1}{n}$. Assume WLOG $m \geqslant n$, Then $\forall \varepsilon > 0$
$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{m-n}{mn} \leqslant \frac{1}{n} < \varepsilon \quad \forall m, n \geqslant N(\varepsilon) = 1 + \left\lceil \frac{1}{\varepsilon} \right\rceil.$$
- $x_n = (-1)^n$ is not a Cauchy sequence, because if $n = 2k$, $m = 2k+1$ for any $k \in \mathbb{N}$, then
$$|x_n - x_m| = |1 - (-1)| = 2.$$
  The definition fails for $\varepsilon = 1$.
- (x_n) on $\mathbb{Q}$ defined by truncation of decimal expansion of $\sqrt{2}$:
$$x_1 = 1, x_2 = 1.4, x_3 = 1.41, x_4 = 1.414, \ldots$$
  This is Cauchy, since for WLOG $m > n$, we have
$$|x_m - x_n| < 10^{-n+1} \to 0 \quad \text{as } n \to \infty.$$
  This sequence does not converge over $\mathbb{Q}$, but it does converge over $\mathbb{R}$.

Exercise. If (x_n) satisfies

$$\forall \varepsilon > 0, \exists N = N(\varepsilon), \forall n \geqslant N : |x_{n+1} - x_n| < \varepsilon,$$

must (x_n) be a Cauchy sequence?

Lemma 1.19

If (x_n) is Cauchy, then it is bounded.

Proof. Take $\varepsilon = 1$. Then $\exists N$ such that $\forall n \geqslant N$,

$$|x_n - x_N| < 1.$$

Hence $|x_n| < 1 + |x_N|$ for all $n \geqslant N$. Note that x_N is a finite number independent of n. So

$$\sup_{n \geqslant 1} |x_n| \leqslant \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |x_N|\}.$$

Lemma 1.20

A complex sequence (x_n) is Cauchy if and only if $(\mathrm{Re}(x_n))$ and $(\mathrm{Im}(x_n))$ are Cauchy in $\mathbb{R}$.

Lemma 1.21

If $x_n \to x$, then (x_n) is Cauchy.

Proof. $\forall \varepsilon > 0, \exists N = N(\varepsilon), \forall m, n \geqslant N$, we have

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &= |x_n - x| + |x - x_m| \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Consider the converse of Lemma 1.21. Note that Example 1.18 (3) shows that there are Cauchy sequences in $\mathbb{Q}$ that do not converge in $\mathbb{Q}$. However, we have the following important theorem.

Theorem 1.22 (Completeness Of $\mathbb{R}$ and $\mathbb{C}$)

Every Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$ converges.

Remark. So one can prove convergence of $\mathbb{R}$ or $\mathbb{C}$ sequences without having to know the actual limit, by showing that they are Cauchy.

Proof. Recall that a sequence on $\mathbb{C}$ is Cauchy/convergent if and only if its real and imaginary parts are Cauchy/convergent. So it suffices to prove the result for real sequences.

We have seen that (x_n) being Cauchy implies that it is bounded by Lemma 1.19. Then by Bolzano-Weierstrass Theorem 1.13, $\exists$ convergent subsequence $\left(x_{n_k}\right)$ with limit $x \in \mathbb{R}$. We have

$$|x_n - x| \leqslant \left|x_n - x_{n_k}\right| + \left|x_{n_k} - x\right|.$$

More precisely, take $\varepsilon > 0$,

- since (x_n) is Cauchy, $\exists N_1 = N_1(\varepsilon)$ such that $\forall m, n \geqslant N_1$, $|x_n - x_m| < \varepsilon$,
- since $x_{n_k} \to x$, $\exists N_2 = N_2(\varepsilon)$ such that $\forall k \geqslant N_2$, $\left|x_{n_k} - x\right| < \varepsilon$.

We can choose $k \geqslant N_2$ such that $n_k \geqslant N_1$, then

$$|x_n - x| < \varepsilon + \varepsilon = 2\varepsilon \quad \forall n \geqslant N_1.$$

1.4 Series and Convergence Tests

Definition 1.23 (Series and Series Convergence)

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence over $\mathbb{R}$ or $\mathbb{C}$. We say that $\sum_{n=1}^{\infty} a_n$ is a **series**.

We say it **converges** if the sequence of partial sums

$$s_k = \sum_{n=1}^{k} a_n$$

converges to some finite $s \in \mathbb{R}$ or $\mathbb{C}$ as $k \to \infty$. In this case, s is called the sum of the series,

$$s = \sum_{n=1}^{\infty} a_n.$$

Example 1.24

- $\sum_{n=1}^{\infty} n$ does not converge as $s_k = \sum_{n=1}^{k} n = \frac{1}{2}k(k+1) \to \infty$ as $k \to \infty$.
- **Geometric series.** $\sum_{k=1}^{\infty} r^n < \infty$ converges iff $|r| < 1$. The partial sums for $|r| < 1$ are

$$s_k = \sum_{n=0}^{k} r^n = \frac{1 - r^{k+1}}{1 - r} \to \frac{1}{1 - r} \quad \text{as } k \to \infty.$$

Lecture 4 · 2026-01-29

- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1, since we have

$$s_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{k+1} \to 1 \quad \text{as } k \to \infty.$$

Lemma 1.25

Fix $\lambda \in \mathbb{C}$. If $\sum a_n$ and $\sum b_n$ converge, then $\sum(\lambda a_n + b_n)$ also converges.

Remark. Note that the product of two convergent series need not converge.

As usual, only the tail of the series matters for convergence.

Lemma 1.26

If $a_n = b_n$ for all $n \geqslant N$ for some $N \in \mathbb{N}$, then $\sum a_n$ converges iff $\sum b_n$ converges.

Proof. Let

$$s_k = \sum_{n=1}^{k} a_n, \quad r_k = \sum_{n=1}^{k} b_n = \sum_{n=1}^{N-1} b_n + \sum_{n=N}^{k} a_n$$
$$= \sum_{n=1}^{N} b_n + \sum_{n=N+1}^{k} a_n$$
$$= s_k + \sum_{n=1}^{N} (b_n - a_n).$$

Note that $\sum_{n=1}^{n}(b_n - a_n)$ is a finite sum, so it does not affect convergence. If $k \to \infty$, then s_k converges iff r_k converges.

Proposition 1.27 (*n*th Term Test)

A necessary condition for $\sum a_n$ to converge is that $a_n \to 0$ as $n \to \infty$.

[*i.e.*, if a_n does not converge to 0, then $\sum a_n$ diverges.]

Remark. $a_n \to 0$ is not a sufficient condition for $\sum a_n$ to converge.

For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though its terms converge to 0. To see that it diverges, note that the partial sums satisfy

$$s_k = \sum_{n=1}^{k} \frac{1}{n}$$

$$\begin{aligned} s_{2k} &= s_k + \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} \\ &\geqslant s_k + k\frac{1}{2k} = s_k + \frac{1}{2}. \end{aligned}$$

Hence (s_k) is not Cauchy, so it diverges.

Proof.

$$\begin{aligned} \sum a_n \text{ converges} &\Leftrightarrow s_k \to s \text{ as } k \to \infty \text{ for some } s \\ &\Rightarrow (s_k) \text{ is Cauchy} \\ &\Rightarrow a_{k+1} = s_{k+1} - s_k \to 0 \text{ as } k \to \infty. \end{aligned}$$

We shall first focus on tests for convergence of $\sum a_n$ where $a_n \geqslant 0$ for all n.

Proposition 1.28 (Comparison Test)

If $0 \leqslant b_n \leqslant a_n$ for all sufficiently large n, then

$$\sum a_n \text{ converges} \Rightarrow \sum b_n \text{ converges.}$$

Proof. Let s_k and r_k be the partial sums of $\sum a_n$ and $\sum b_n$. Because $a_n, b_n \geqslant 0$, the sequences (s_k) and (r_k) are increasing. Since $s_k \to s$, we have $s_k \leqslant s$ for all k. Hence

$$\begin{aligned} b_n \leqslant a_n &\Rightarrow \sum_{n=1}^{k} b_n \leqslant \sum_{n=1}^{k} a_n \leqslant s \\ &\Rightarrow r_k \text{ is bounded above by } s \\ &\Rightarrow (r_k) \text{ converges by monotone convergence theorem.} \end{aligned}$$

Example 1.29

$\sum \frac{1}{n^2}$ converges. This is because

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leqslant 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \leqslant 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

so by Comparison Test 1.28, $\sum \frac{1}{n^2}$ converges.

The next two tests are about asymptotic comparisons to the geometric series.

Proposition 1.30 (Root Test)

If $a_n \geqslant 0$ for all n, then consider $\sqrt[n]{a_n}$, and assume $\exists a$ such that $a = \lim_{n \to \infty} \sqrt[n]{a_n}$.

Then

- $a < 1$ implies $\sum a_n$ converges.
- $a > 1$ implies $\sum a_n$ diverges.
- $a = 1$ is inconclusive.

Proof. If $a > 1$, then by the definition of limit,

$$\exists N \in \mathbb{N}, \forall n \geqslant N : a_n^{\frac{1}{n}} > 1.$$

This implies that $a_n > 1$ for all $n \geqslant N$, so $\sum a_n$ diverges by nth term test 1.27.

Now, if $a < 1$, then there is some $r \in \mathbb{R}$ such that $a < r < 1$. By the definition of limit,

$$\exists N \in \mathbb{N}, \forall n \geqslant N, a_n^{\frac{1}{n}} < r.$$

Hence $a_n \leqslant r^n$ for all $n \geqslant N$. By Comparison Test 1.28, $\sum a_n$ converges since $\sum r^n$ converges.

Example 1.31

- $\sum \frac{1}{2^n}$: $\sqrt[n]{a_n} = \frac{1}{2} < 1$, so it converges.
- $\sum 4^n$: $\sqrt[n]{a_n} = 4 > 1$, so it diverges.

Proposition 1.32 (Ratio Test)

If $a_n \geqslant 0$ for all n, then consider $\frac{a_{n+1}}{a_n}$, and assume $\exists a$ such that $a = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.

Then

- $a < 1$ implies $\sum a_n$ converges.
- $a > 1$ implies $\sum a_n$ diverges.
- $a = 1$ is inconclusive.

Example 1.33

- $\sum \frac{1}{n}$ (divergent) and $\sum \frac{1}{n^2}$ (convergent) are both inconclusive under the root and ratio tests, since both have limit 1 in both tests.
- $\sum \frac{n}{2^n}$ converges, since

  $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1.$$

  Or alternatively, by the root test,

  $$\left(\frac{n}{2^n}\right)^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{2} \to \frac{1}{2} < 1.$$

  > ***Remark.*** To show that $n^{\frac{1}{n}} \to 1$, write $n^{\frac{1}{n}} = e^{\frac{\log n}{n}}$ and L'Hospital's rule shows that $\frac{\log n}{n} \to 0$.

Exercise. Show that if the ratio test is inconclusive, then so is the root test. Show also that the converse is not true, using

$$a_n = \begin{cases} 2^{-n} & n \text{ even} \\ 2^{-(n+1)} & n \text{ odd} \end{cases}$$

Lecture 5 · 2026-01-31

Proposition 1.34 (Integral Test)

Suppose $f : [1,\infty) \to [0,\infty)$ is a continuous decreasing function (so it is integrable in $[1,N]$ for each $N \in \mathbb{N}$ [we will see this later]). Let $a_n = f(n)$ for each $n \in \mathbb{N}$.

Then

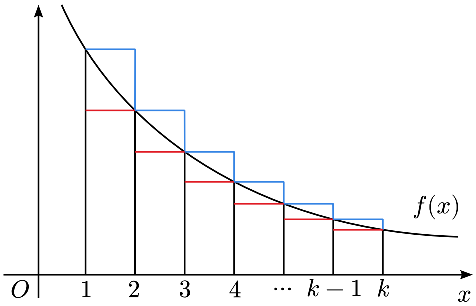
$$\sum_{n\geqslant 1} a_n \text{ converges} \Leftrightarrow \lim_{n\to\infty} f(t)\,\mathrm{d}t \text{ exists.}$$

Furthermore, as $k \to \infty$,

$$\sum_{n=1}^{k} a_n - \int_1^k f(x)\,\mathrm{d}x \to l \text{ for some } l \in [0, f(1)].$$

> ***Remark.*** The RHS is an improper integral, which will be discussed later. The last part tells us that the integral is a good approximation for the series (if it converges), or the rate of divergence (if it diverges).

Proof.



We have

$$s_k - a_1 = \sum_{n=2}^{k} a_n = \sum \textcolor{red}{\square} \leqslant \int_1^k f(t)\,\mathrm{d}t,$$

$$s_{k-1} = \sum_{n=1}^{k-1} a_n = \sum \textcolor{blue}{\square} \geqslant \int_1^k f(t)\,\mathrm{d}t.$$

$[\Rightarrow]$ $\int_1^k f(t)\,\mathrm{d}t \leqslant s_{k-1} \leqslant s$ since (s_k) is increasing and converges by assumption. Thus $\left(\int_1^k f(t)\,\mathrm{d}t\right)_k$ is increasing and bounded above, so it converges.

$[\Leftarrow]$ if the integral exists, then $\left(\int_1^k f(t)\,\mathrm{d}t\right)_k$ is bounded. Hence (s_k) is a monotone bounded sequence and it converges.

For the last part, let $b_k = \sum_{n=1}^k a_n - \int_1^k f(x)\,\mathrm{d}x$. We have

- $b_k - b_{k-1} = a_k - \int_{k-1}^k f(t)\,\mathrm{d}t = f(k) - \int_{k-1}^k f(t)\,\mathrm{d}t \leqslant 0.$
- $b_k \geqslant a_k = f(k) \geqslant 0.$

Hence (k) is decreasing and bounded below, so it converges to some $l \geqslant 0$. Also, since

$$0 \leqslant f(k) \leqslant b_k \leqslant a_1 = f(1)$$

we get $0 \leqslant l \leqslant f(1)$.

Example 1.35

- $\sum \frac{1}{n^p}$ converges iff $p > 1$.

  Note that

$$\lim_{x\to\infty} \int_1^x \frac{1}{t^p}\,\mathrm{d}t = \lim_{x\to\infty} \begin{cases} (1-p)x^{-p} & p \neq 1 \\ \log x & p = 1 \end{cases} + \text{constant}$$

  which exists for $p > 1$ and diverges for $p \leqslant 1$.

  > ***Remark.*** This is a much easier way to see the divergence of the harmonic series. Note *a posteriori* that the divergence is not surprising, since for $\sum a_n$ to converge we need $a_n \to 0$ sufficiently fast to overcome the growth in the number of terms we are adding up.
  >
  > Rough calculation suggests that $a_n \ll \frac{1}{n}$ for large n would be enough for convergence.

- $\sum \frac{1}{n \log n}$ diverges since

$$\int \frac{1}{t\log t}\,\mathrm{d}t = \int \frac{1}{u}\,\mathrm{d}u = \log u + C = \log(\log t) + C$$

with the substitution $u = \log t$.

- $\sum \frac{1}{n\log^2 n}$ converges since

$$\int \frac{1}{t\log^2 t}\,\mathrm{d}t = \int \frac{1}{u^2}\,\mathrm{d}u = -\frac{1}{u} + C = -\frac{1}{\log t} + C$$

with the substitution $u = \log t$.

Proposition 1.36 (Cauchy Condensation Test [Non-Examinable])

Let $a_n \geqslant 0$ for all n, and suppose that (a_n) is decreasing. Then

$$\sum a_n \text{ converges} \Leftrightarrow \sum 2^n a_{2^n} \text{ converges.}$$

Proof. We have

$$\int_1^x f(t)\,\mathrm{d}t = \log 2 \int_0^{2^x} f(2^t)2^t\,\mathrm{d}t$$

using the substitution $u = 2^t\,\mathrm{d}t$.

From Integral Test 1.34, we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \text{ converges} &\Leftrightarrow \lim_{x\to\infty}\int_1^x f(t)\,\mathrm{d}t \text{ exists} \\ &\Leftrightarrow \lim_{y\to\infty}\int_0^y f(2^t)2^t\,\mathrm{d}t \text{ exists.}\end{aligned}$$

Hence, letting $g(t) = 2^t f(2^t)$,

$$\begin{aligned} f \text{ decreasing} &\Rightarrow f\left(2^{k+1}\right) \leqslant f(t) \leqslant f\left(2^k\right) && \forall t \in [k, k+1], \\ &\Rightarrow 2^k f\left(2^{k+1}\right) \leqslant g(t) \leqslant 2^{k+1} f\left(2^k\right) && \forall t \in [k, k+1], \\ &\Rightarrow \frac{1}{2}\sum_{n=1}^{k+1} 2^{n+1}a_{2^{n+1}} \leqslant \int_1^{k+1} g(t)\,\mathrm{d}t \leqslant 2\sum_{n=1}^{k+1} 2^n a_{2^n}. \end{aligned}$$

Thus, $\sum 2^n a_{2^n}$ converges iff $\lim_{y\to\infty}\int_0^y f(2^t)2^t\,\mathrm{d}t$ exists, and the result follows.

Proposition 1.37 (Alternating Series Test)

Let (a_n) be a decreasing sequence with $a_n \geqslant 0$ and $a_n \to 0$. Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Example 1.38

> $\sum \frac{(-1)^{n+1}}{n}$ converges though $\sum \frac{1}{n}$ diverges, by the alternating series test. [In Section 5, we will show that it converges to $\log 2$.]

Lecture 6 · 2026-02-03

Proof. Let $s_k = \sum_{n=1}^{k} (-1)^{n+1} a_n$ be the partial sums. Note that

$$s_{2k} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2k-1} - a_{2k})$$

Hence,

$$s_2 \leqslant s_4 \leqslant s_6 \leqslant \ldots \leqslant s_{2k} \leqslant \ldots$$

so (s_{2k}) is an increasing sequence.

Also,

$$s_{2k+1} = a_1 + (-a_2 + a_3) + \ldots + (-a_{2k} + a_{2k+1})$$

so

$$s_1 \geqslant s_3 \geqslant s_5 \geqslant \ldots \geqslant s_{2k+1} \geqslant \ldots$$

and (s_{2k+1} is decreasing.

Moreover, $s_{2k+1} - s_{2k} = a_{2k+1} \geqslant 0$, so we have

$$a_1 - a_2 = s_2 \leqslant s_{2k} \leqslant s_{2k+1} \leqslant s_1 = a_1.$$

Therefore, both (s_{2k}) and (s_{2k+1}) are bounded. By Monotone Convergence Theorem 1.12, both sequences converge. Let $s_{2k} \to s$ and $s_{2k+1} \to \tilde{s}$ as $k \to \infty$.

Note that

$$\tilde{s} - s = \lim_{k \to \infty} (s_{2k+1} - s_{2k}) = \lim_{k \to \infty} a_{2k+1} = 0.$$

Lemma 1.39

If the odd and even subsequences of a sequence both converge to the same limit, then the whole sequence converges to the same limit.

Proof. Forall $\varepsilon > 0$,

$$\exists N_1 = N_1(\varepsilon), \forall k \geqslant N_1 : |s_{2k} - s| < \varepsilon,$$
$$\exists N_2 = N_2(\varepsilon), \forall k \geqslant N_2 : |s_{2k+1} - s| < \varepsilon.$$

Hence $|s_n - s| < \varepsilon$ for all $n \geqslant \max\{2N_1, 2N_2 + 1\}$.

Therefore, $s_k \to s$ as $k \to \infty$.

Proposition 1.40 (Dirichlet Test)

Let (a_n) be a decreasing sequence with $a_n \geqslant 0$ and $a_n \to 0$. Let (b_n) be a sequence such that the sequence of partial sums $\left(s_k = \sum_{n=1}^{k} b_n\right)_k$ is bounded.

Then $\sum a_n b_n$ converges.

Definition 1.41 (Absolute Convergence)

A series $\sum a_n$ is said to **converge absolutely** if $\sum |a_n|$ converges.

Remark. $|a_n| \geqslant 0$ for all n, hence root, ratio, *etc.* tests can be applied to test for absolute convergence.

Lemma 1.42

If $\sum a_n$ converges absolutely, then it converges.

Remark. The converse is not true by Example 1.38. Hence, absolute convergence is a strictly stronger notion. We call series that converge but not absolutely **conditionally convergent**.

Proof. Let $s_k = \sum_{n \leqslant k} a_n$, and let $r_k = \sum_{n \leqslant k} |a_n|$. Since r_k is convergent, it is Cauchy.

$$|s_k - s_l| = \left|\sum_{n=l+1}^{k} a_n\right| \leqslant \sum_{n=l+1}^{k} |a_n| = |r_k - r_l| < \varepsilon \quad \forall k, l \geqslant N(\varepsilon).$$

Thus, (s_k) is Cauchy and hence convergent.

Conditionally convergent series can behave badly under rearrangements.

Example 1.43

Consider

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Rearrange terms on the right as

$$\begin{aligned} &\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \dots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right) \\ &= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}. \end{aligned}$$

This suggests that the rearranged series sums to half the original series.

> In a conditionally convergent series, the order of the sum matters. This is not the case for absolutely convergent series, where any rearrangement converges to the same sum.

Proposition 1.44 (Rearrangements of Absolutely Convergent Series)

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Let $a'_n = a_{\sigma(n)}$. Then if $\sum a_n$ is absolutely convergent, we have

$$\sum a'_n = \sum a_n.$$

Proof. Let $s_k = \sum_{n \leqslant k} a_n$. By assumption, $\exists s$ such that $s_k \to s$ as $k \to \infty$. For $\varepsilon > 0$,

$$\exists N = N(\varepsilon), \forall k \geqslant N : |s_k - s| < \varepsilon \quad \text{and} \quad \sum_{k \geqslant N} |a_k| < \varepsilon.$$

Now, since σ is a bijection, $\exists M \geqslant N$ such that $a_1, \ldots, a_N$ is contained in $a'_1, \ldots, a'_M$. Hence, for $m \geqslant M$,

$$\sum_{n=1}^{m} a'_n = \underbrace{\sum_{n=1}^{N} a_n}_{s_N} + \sum_{n=N+1}^{m} a'_n$$

$$\begin{aligned} \left| \sum_{n=1}^{m} a'_n - s \right| &= |s_N - s| + \sum_{k \geqslant N} |a_k| \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Hence $\sum a'_n = \sum a_n$.

2 Continuity of Functions

2.1 Limits of Functions

Take a function $f : X \to \mathbb{C}$ with $X \subseteq \mathbb{C}$. Consider the meaning of $f(z) \to y$ as $z \to a$, even if $a \notin X$.

A classic example is $\frac{\sin x}{x}$ at $a = 0$, which is not in the domain of the function. The reason we may think of $\lim_{x \to 0} \frac{\sin x}{x}$ is that there are points in the domain that are very close to 0. In other words, 0 is an accumulation point for $\mathbb{R} \setminus \{0\}$, since for any threshold $\delta > 0$, there are points in $\mathbb{R} \setminus \{0\}$ within δ of 0.

Definition 2.1 (Accumulation Point)

Let $X \subseteq \mathbb{C}$, and $a \in \mathbb{C}$. We say that a is an **accumulation point** for X if

$$\forall \delta > 0, \exists z \in X \setminus \{a\} : |z - a| < \delta.$$

If $a \in X$ and is not an accumulation point for X, we say that a is an **isolated point** of X.

Example 2.2

- For $X = \{0\} \cup [1, 2]$, 0 is an isolated point of X, while any point in $[1, 2]$ is an accumulation point of X.

Lecture 7 · 2026-02-05

- The points on the circle $S = \{z \in \mathbb{C} : |z| = 1\}$ are accumulation points for $D = \{z \in \mathbb{C} : |z| < 1\}$.
- All the points in $D = \{z \in \mathbb{C} : |z| < 1\}$ are accumulation points for D.
- For the set $\{|z| \leqslant 1\}$, then all the points are accumulation points.

Lemma 2.3

Let $X \subseteq C$, $a \in \mathbb{C}$. Then a is a accumulation point iff there exists (x_n) in $X \setminus \{a\}$ such that $x_n \to a$ as $n \to \infty$.

Hence, for $f(z) \to y$ as $z \to a$ to be meaningful, we need $a \in X$, or if $a \notin X$, it should be an accumulation point for X. The rationale for our definition are

- for any threshold $\varepsilon > 0$ (no matter how small it is), there exist points in $\operatorname{Im} f$ which are ε-close to y, *i.e.*

$$\{z \in X : |f(z) - y| < \varepsilon\} \neq \varnothing.$$

- furthermore, it must contain all points in X that are sufficiently close to a, *i.e.*

$$\exists \delta > 0 : \forall z \in X, |z - a| < \delta \Rightarrow |f(z) - y| < \varepsilon.$$

Definition 2.4 (Limit of a Function)

Let $f : X \subseteq \mathbb{C} \to \mathbb{C}$. Take $a \in \mathbb{C}$ such that a is an accumulation point for X. We say that $f(z) \to y$ as $z \to a$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall z \in X : 0 < |z - a| < \delta \Rightarrow |f(z) - y| < \varepsilon.$$

y is called the **limit** of f as $z \to a$, and we write $\lim_{z \to a} f(z) = y$.

In particular, for $a \in \mathbb{C}$ where a is an accumulation point for X, we say that f diverges (to ∞) as $z \to a$ if $\frac{f(z)}{|f(z)|} \to l$ for some $l \in \mathbb{C}$, and

$$\forall L > 0, \exists \delta > 0, \forall z \in X : 0 < |z - a| < \delta \Rightarrow |f(z)| > L.$$

Remark. If a is an isolated point of X, then we can always find ε sufficiently small that

$$|z - a| < \varepsilon, z \in X \Leftrightarrow z = a$$

So the definition of limit can be made for isolated points, but it is not very interesting.

Example 2.5

$\frac{\sin x}{x}$ has fomain $\mathbb{R} \setminus \{0\}$. Consider its limit as $x \to 0$.

Claim.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Proof. Recall that there is a geometric argument using trignometric circle that shows $\cos x < \frac{\sin x}{x} < 1$ for all $x \in \left(0, \frac{\pi}{2}\right)$. Hence

$$\left|\frac{\sin x}{x} - 1\right| < 1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right) < \frac{x^2}{2}.$$

Hence $\forall \varepsilon > 0$, choosing $\delta(\varepsilon) = \sqrt{2\varepsilon}$ gives

$$|x - 0| < \delta \Rightarrow \left|\frac{\sin x}{x} - 1\right| < \varepsilon.$$

We can also give a sequential characterisation of limits, which is often easier to work with.

Lemma 2.6 (Sequential Characterisation of Limits)

Let $f : X \subseteq \mathbb{C} \to \mathbb{C}$. Let $a \in \mathbb{C}$ where a is an accumulation point of X. Then

$$\begin{aligned} f(z) \to y \text{ as } z \to a \quad &\Leftrightarrow \quad f(z_n) \to y \text{ for every } (z_n) \text{ on } X. \\ f(z) \text{ diverges as } z \to a \quad &\Leftrightarrow \quad f(z_n) \text{ diverges for every } (z_n) \text{ on } X. \end{aligned}$$

with $(z_n) \to a$ and z_n is not the constant sequence.

Lemma 2.7

Suppose $f : X \subseteq \mathbb{C} \to \mathbb{C}$, and it has a limit at a. Then the limit is unique.

Proof. Suppose $f(z) \to y$ as $z \to a$, and $f(z) \to x$ as $z \to a$ are both true.

We have

$$|x - y| = |x - f(z) + f(z) - y| \leqslant |x - f(z)| + |f(z) - y|.$$

Note that LHS does not depend on z, but RHS does. So taking limit as $z \to a$ gives

$$|x - y| \leqslant \lim_{z \to a} |x - f(z)| + \lim_{z \to a} |f(z) - y| = 0.$$

Hence $x = y$.

Lemma 2.8

Let $f : X \subseteq \mathbb{C} \to \mathbb{C}$. Let $a \in \mathbb{C}$ where a is an accumulation point of X. Suppose $\lim_{z \to a} f(z) = y$, $\lim_{z - a} g(z) = x$. Then,

$$\lim_{z \to a}(f(z) + g(z)) = y + x, \quad \lim_{z \to a}(f(z)g(z)) = yx, \quad \lim_{z \to a}\left(\frac{f(z)}{g(z)}\right) = \frac{y}{x} \text{ if } \forall z \in X,\ g(z) \neq 0 \text{ and } x \neq 0.$$

2.2 Continuity of Functions

From the previous section, we can compute $\lim_{z \to a} f(z)$ if a is a accumulation point for X.

Definition 2.9 (Continuity of a Function)

Let $f : X \subseteq \mathbb{C} \to \mathbb{C}$. We say that f is **continuous** at every point in X.

Take $a \in X$. We say that f is continuous at a if

$$\forall \varepsilon > 0, \exists \delta \in \mathbb{R}, \forall z \in X : |z - a| < \delta \Rightarrow |f(z) - f(a)| < \varepsilon.$$

Remark. If $a \in X$ is an isolated point, then f is continuous at a. Inf $a \in \mathbb{C}$ is an accumulation point for X, then f is continuous at a iff $\lim_{z \to a} f(z) = f(a)$.

Example 2.10

- $f(z) = z$ is continuous, since $\forall \varepsilon > 0, \exists \delta$ given by $\delta(\varepsilon) = \varepsilon$ such that $\forall z \in X, |z - a| < \delta \Rightarrow |f(z) - f(a)| = |z - a| < \varepsilon$.
- $f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not continuous at 0, since $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Lecture 8 · 2026-02-10

Definition 2.11 (Sequential Continuity)

Let $f : X \subseteq \mathbb{C} \to \mathbb{C}$. We say that f is **sequentially continuous** at $a \in X$ if for every sequence (z_n) in X such that $z_n \to a$, we have $f(z_n) \to f(a)$.

Proposition 2.12

Let $f : X \subseteq \mathbb{C} \to \mathbb{C}$. Then f is continuous at $a \in X$ iff f is sequentially continuous at a.

Proof.

[$\Rightarrow$] By continuity of f, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon)$ such that $|z - a| < \delta \Rightarrow |f(z) - f(a)| < \varepsilon$. Take (z_n) on X with $z_n \to a$, then $\exists N = N(\varepsilon)$ such that $|z_n - a| < \delta$ for all $n \geqslant N$. Hence $|f(z_n) - f(a)| < \varepsilon$ for all $n \geqslant N$, so $f(z_n) \to f(a)$.

[$\Leftarrow$] We shall prove by contradiction. Suppose f is not continuous at a, then $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists z \in X$ with $|z - a| < \delta$ but $|f(z) - f(a)| \geqslant \varepsilon$.

Take $\delta = 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$ to find (z_n) with $|z_n - a| < \frac{1}{n} \to 0$ but $|f(z_n) - f(a)| \geqslant \varepsilon$ for all n. Hence $z_n \to a$ but $f(z_n)$ does not converge to $f(a)$, contradicting the sequential continuity of f at a.

Example 2.13

- For

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

  consider the sequence (x_n) with $x_n = \frac{1}{(2n+\frac{1}{2})\pi}$. Then $f(x_n) = 1$ for all n and $x_n \to 0$. Hence f is not sequentially continuous at 0, and thus not continuous at 0.

- Consider $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$. f is discontinuous at every point $a \in \mathbb{R}$:
  - if $a \in \mathbb{Q}$ then $\exists (x_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$ with $x_n \to a$ but $f(x_n) = 0$ for all n and $f(a) = 1$.
  - if $a \in \mathbb{R} \setminus \mathbb{Q}$ then $\exists (x_n) \subseteq \mathbb{Q}$ with $x_n \to a$ but $f(x_n) = 1$ for all n and $f(a) = 0$.

- Consider $f(x) = \sin x$, we shall show that it is continuous at every point $a \in \mathbb{R}$. Fix $a \in \mathbb{R}$. We can choose $\delta(\varepsilon) = \min\left(\varepsilon, \frac{\pi}{2}\right)$,

$$\begin{aligned} |f(x) - f(a)| &= |\sin x - \sin a| \\ &\leqslant 2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right) \\ &\leqslant \left|\sin\left(\frac{x-a}{2}\right)\right| \\ &\leqslant |x - a| \quad \text{by taking } \frac{x-a}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned}$$

  Therefore, $|f(x_n) - f(a)| \leqslant |x_n - a|$ holds for n sufficiently large.

Lemma 2.14

Let $f, g : X \subseteq \mathbb{C} \to \mathbb{C}$ be continuous at $a \in X$. Then so are $\lambda f + g$ and fg. If $f(z) \neq 0$ forall $z \in X$, then so is $\frac{1}{f}$.

Lemma 2.15

Let $U, V \in \mathbb{C}$, $f : U \to V$, $g : V \to \mathbb{C}$. If f is continuous at $a \in U$ and g is continuous at $f(a) \in V$, then $g \circ f : U \to \mathbb{C}$ is continuous at a.

Proof. Since g is continuous at $f(a)$,

$$\forall \varepsilon > 0, \exists \sigma = \sigma(\varepsilon), \forall y \in V : |y - f(a)| < \sigma \Rightarrow |g(y) - g(f(a))| < \varepsilon.$$

Also, f is continuous at a. So

$$\exists \delta = \delta(\varepsilon), \forall z \in U : |z - a| < \delta \Rightarrow |f(z) - f(a)| < \sigma.$$

Putting everything together,

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon), \forall z \in U : |z - a| < \delta \Rightarrow |f(z) - f(a)| < \sigma \Rightarrow |g(f(z)) - g(f(a))| < \varepsilon.$$

Hence $g \circ f$ is continuous at a.

Remark. Composition preserves continuity.

2.3 Extreme Value Theorem

Definition 2.16 (Closed Set)

A set $X \subseteq \mathbb{C}$ is **closed** if all sequences (x_n) in X which converge in $\mathbb{C}$ have their limits in X.

Example 2.17

- $[0, 1]$ is closed,
- $(0, 1)$ is not closed,
- $(0, 1]$ or $[0, 1)$ are not closed,
- $[0, \infty)$ is closed,
- $(0, \infty)$ is not closed.

Definition 2.18 (Bounded Set)

We say $X \subseteq \mathbb{C}$ is bounded if $\exists M > 0$ such that $X \subseteq \{z \in \mathbb{C} : |z| \leqslant M\}$.

In other words, if $\exists M > 0$ such that $\sup_{z \in X} |z| \leqslant M$.

Example 2.19

- $[0, 1]$, $(0, 1)$, $[0, 1)$ are bounded,

- $[0, \infty)$, $(0, \infty)$ are not bounded.

Proposition 2.20 (Continuity Preserves Closedness and Boundedness)

Let $X \subseteq \mathbb{C}$ be a closed bounded set. If $f : X \to \mathbb{C}$ is continuous then $f(X) \subseteq \mathbb{C}$ is a bounded closed subset of $\mathbb{C}$.

Lecture 9 · 2026-02-12

Proof.

[$f(X)$ is bounded.] Suppose $f(X)$ is not bounded, then for each $n \in \mathbb{N}$ we can find x_n such that $|f(x_n)| > n$. Now, (x_n) is a sequence in X, and it must be bounded since X is bounded.

By Bolzano-Weierstrass Theorem 1.13, there exists a convergent subsequence $\left(x_{n_k}\right)$. Let the limit of $\left(x_{n_k}\right)$ be x. Since X is closed, $x \in X$. On the other hand, $\left|f\left(x_{n_k}\right)\right| > n_k \geqslant k$. So $f\left(x_{n_k}\right)$ cannot converge as $k \to \infty$. Then f is not continuous at x. ※ Therefore $f(X)$ must be bounded.

[$f(X)$ is closed.] Take (y_n) in $f(X)$, suppose that it converges to some $y \in \mathbb{C}$. We want to show that $y \in f(X)$. Note that $y_n \in f(X) \Rightarrow \exists x_n \in X$ such that $f(x_n) = y_n$. This sequence (x_n) is inside X, hence it is bounded. Copying the argument above to get $x_{n_k} \to x \in X$. Thus $y_{n_k} = f\left(x_{n_k}\right)$. By continuity of f,

$$y = \lim_{k \to \infty} y_{n_k} = \lim_{k \to \infty} f\left(x_{n_k}\right) = f\left(\lim_{k \to \infty} x_{n_k}\right) = f(x).$$

Hence $y \in f(X)$.

Theorem 2.21 (Extreme Value Theorem)

Let $X \subseteq \mathbb{R}$ be a closed bounded set. If $f : X \to \mathbb{R}$ is continuous, then there exist $a, b \in X$ with

$$a = \sup f(X), \quad b = \inf f(X).$$

Remark. We know by Supremum Axiom (from IA Numbers and Sets), that there exist supremum and infimum of $f(X)$. The proof here is that these are always attained, so that they actually maximum and minimum.

Proof. We will focus on the first equality, since the other can be proved similarly.

Let $M = \sup f(X)$. Hence, $M - \frac{1}{n}$ is not an upper bound for $f(X)$ for all $n \in \mathbb{N}$. Hence, we can find a sequence (y_n) on $f(X)$ such that

$$M - \frac{1}{n} < y_n \leqslant M.$$

Note that $\exists x_n$ such that $y_n = f(x_n)$ for each of y_n. This gives us a sequence (x_n) on X such that $f(x_n) = y_n$ for all n with $M - \frac{1}{n} < f(x_n) \leqslant M$. Now take limits and use the fact that f is continuous, we get

$$M \leqslant \lim_{n\to\infty} f(x_n) \leqslant M \Rightarrow \lim_{n\to\infty} f(x_n) = M.$$

By closedness of X, $f(X)$ is also closed by Proposition 2.20. Hence $M \in f(X)$, so there exists $a \in X$ such that $f(a) = M$. Hence $a = \sup f(X)$.

2.4 Intermediate Value Theorem

Theorem 2.22 (Intermediate Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is an interval. Hence, if $f(a) \leqslant y \leqslant f(b)$, then there exists $c \in [a, b]$ such that $f(c) = y$.

Remark. The theorem guarantees existence, but not uniqueness.

Proof. If f is constant, or if $y = f(a)$ or $y = f(b)$, then this is trivially true.

If not, WLOG assume $f(a) < y < f(b)$, and let $S = \{x \in [a, b] : f(x) \leqslant y\}$. Then $a \in S$, so S is non-empty. Also, S is bonunded above by b, so $\exists d = \sup S \in [a, b]$. We aim to show that $f(d) = y$.

- Suppose $f(d) > y$. Then $\varepsilon = f(d) - y > 0$. By continuity of f, $\exists \delta = \delta(\varepsilon) > 0$ such that $\forall |x - d| < \delta$, $x \in [a, b] \Rightarrow |f(x) - f(d)| < \varepsilon \Rightarrow f(x) > f(d) - \varepsilon = y \Rightarrow x \notin S$.

  This means that $(d - \delta, d) \cap S = \varnothing$, which contradicts the definition of d.
- Suppose $f(d) < y$. Then $\varepsilon = y - f(d) > 0$. By continuity of f, $\exists \delta = \delta(\varepsilon) > 0$ such that $\forall |x - d| < \delta$, $x \in [a, b] \Rightarrow |f(x) - f(d)| < \varepsilon \Rightarrow f(x) < f(d) + \varepsilon = y$. But then, $f\left(d + \frac{\delta}{2}\right) < y \Rightarrow d \leqslant d + \frac{\delta}{2} \in S$, again contradicting the definition of d.

Hence $f(d) = y$.

Example 2.23

We can apply this to show the existence of N-th roots (where $N \in \mathbb{N}$). Take $t > 0$, and consider

$$\begin{aligned} f : [0, 1 + t] &\to \mathbb{R} \\ x &\mapsto x^N. \end{aligned}$$

This is a continuous function, and $0 = f(0) < t < (1 + t)^N = f(1 + t)$. By Intermediate Value Theorem 2.22, $\exists c \in [0, 1 + t]$ such that $f(c) = t \Leftrightarrow c^N = t$, Hence c is a positive N-th root of t.

Definition 2.24 (Monotone Function)

Consider $f : [a, b] \to \mathbb{R}$. We say f is (strictly) monotone if either

- it is (strictly) increasing, so $a \leqslant x_1 \leqslant x_2 \leqslant b \Rightarrow f(x_1) \leqslant f(x_2)$,
- it is (strictly) decreasing, so $a \leqslant x_1 \leqslant x_2 \leqslant b \Rightarrow f(x_1) \geqslant f(x_2)$.

Proposition 2.25 (Inverse Function Theorem, Version 1)

> Let $f : [a, b] \to \mathbb{R}$ be a continuous function that is strictly monotone. Let $c = \min\{f(a), f(b)\}$ and $d = \max\{f(a), f(b)\}$. Then $f : [a, b] \to [c, d]$ is a bijection and $f^{-1} : [c, d] \to [a, b]$ is continuous and strictly monotone.

Lecture 10 · 2026-02-14

Proof.

[Bijiectivity of f.] Since f is continuous and monotone, so it maps $[a, b]$ to $[c, d]$ because its monotonicity implies that the extreme values are attained at the endpoints. Since it is strictly monotone, it is injective. By Intermediate Value Theorem 2.22, it is surjective. Hence f is a bijection.

[Monotonicity of f^{-1}.] WLOG take f to be strictly increasing. If $f^{-1} : [c, d] \to [a, b]$ is not strictly increasing, then $\exists y_1, y_2 \in [c, d]$ such that $y_1 < y_2$ yet $f^{-1}(y_1) \geqslant f^{-1}(y_2)$. Since f is strictly increasing, $f\left(f^{-1}(y_1)\right) \geqslant f\left(f^{-1}(y_2)\right) \Rightarrow y_1 \geqslant y_2$, contradicting the choice of y_1, y_2. Hence f^{-1} is strictly increasing.

[Continuity of f^{-1}.] Fix $y_0 \in [c, d]$ and $x_0 = f^{-1}(y_0)$. There are three cases:

- if $x_0 \in (a, b)$, fix $\varepsilon > 0$. Then choose $\eta \in (0, \varepsilon]$ sufficiently small that $[x_0 - \eta, x_0 + \eta] \subseteq [a, b]$. Now, f is strictly increasing, so
$$f(x_0 - \eta) < f(x_0) = y_0 < f(x_0 + \eta).$$
Then take $\delta = \min\{f(x_0 + \eta) - y_0, y_0 - f(x_0 - \eta)\}$. We want to prove that for $y \in [c, d]$,
$$|y - y_0| < \delta \Rightarrow \left|f^{-1}(y) - x_0\right| < \varepsilon.$$
We have
$$\begin{aligned} |y - y_0| < \delta &\Rightarrow y_0 - \delta < y < y_0 + \delta \\ &\Rightarrow f(x_0 - \eta) < y < f(x_0 + \eta) \\ &\Rightarrow x_0 - \eta < f^{-1}(y) < x_0 + \eta \\ &\Rightarrow \left|f^{-1}(y) - x_0\right| < \eta \leqslant \varepsilon. \end{aligned}$$
- if $x_0 = a$, fix $\varepsilon > 0$. Then $y_0 = f(a)$. Choose $\eta = \min\{\varepsilon, b - a\}$ and set $\delta = f(a + \eta) - f(a) > 0$. Then,
$$\begin{aligned} |y - y_0| < \delta, y \in [c, d] &\Rightarrow f(a) \leqslant y \leqslant f(a + \eta) \\ &\Rightarrow a \leqslant f^{-1}(y) \leqslant a + \eta \\ &\Rightarrow \left|f^{-1}(y) - a\right| \leqslant \eta \leqslant \varepsilon. \end{aligned}$$
- if $x_0 = b$, the case is similar to the previous one.

3 Differentiation

3.1 Introduction

Definition 3.1 (Differentiability and Derivative)

Let $f : X \subseteq \mathbb{C} \to \mathbb{C}$ with $a \in X$. We say that f is **differentiable** at a if the limit

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists. [We require $a+h \in X$.]

This limit is called the **derivative** of f at a, and is denoted by $f'(a)$ or $\frac{\mathrm{df}}{\mathrm{dx}}(a)$.

Remark. At isolated points, the definition of derivative is meaningless. For accumulation points, we can distinguish between

- interior points, where we can approach a from both sides, and we need the limit to be direction-independent, and
- non-interior points, where the limit is domain-restricted.

Example 3.2

- $f(z) = z$ is differentiable at every point. This is because we have

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{z+h-z}{h} = \lim_{h\to 0} 1 = 1.$$

- $f(z) = \overline{z}$ is not differentiable at any point. This is because we have

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{\overline{z+h}-\overline{z}}{h} = \lim_{h\to 0} \frac{\overline{h}}{h}.$$

  If this limit exists, then it must be the same as the limit along the real axis, which is 1, and the limit along the imaginary axis, which is -1.

- $f(x) = \sin x$ on $\mathbb{R}$ is differentiable at every point. We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \sin \frac{h}{h}. \end{aligned}$$

> We can use results from lectures that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0} \sin \frac{h}{h} = 1$ to conclude that $f'(x) = \cos x$.

We can also derive some properties of derivatives from properties of limits.

Lemma 3.3

let $f, g : X \in \mathbb{C} \to \mathbb{C}$ be differentiable at $a \in X$. Then so are $f + g, fg$, and $\frac{1}{f}$ if $f(z) \neq 0$ for all $z \in X$.

Moreover, we have

$$\begin{aligned}(f+g)'(a) &= f'(a) + g'(a)\\ (fg)'(a) &= f'(a)g(a) + f(a)g'(a)\\ \left(\frac{1}{f}\right)'(a) &= -\frac{f'(a)}{[f(a)]^2}.\end{aligned}$$

Proof. These follow from last chapter. The addition rule is left as an exercise. We have

$$\begin{aligned}(fg)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}\\ &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h}\\ &= \lim_{h\to 0} \frac{(f(a+h) - f(a))g(a+h) + f(a)(g(a+h) - g(a))}{h}\\ &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} g(a+h) + f(a) \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}\\ &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \lim_{h\to 0} g(a+h) + f(a) \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}\\ &= f'(a)g(a) + f(a)g'(a).\end{aligned}$$

For the last step, we need to show that a function is continuous at a if it is differentiable at a. This will be dealt with later.

The reciprocal rule is left as an exercise.

Example 3.4

- $f(z) = z^2$ is differentiable with $f'(z) = 2z$. By induction, we can show that $f(z) = z^n$ is differentiable with $f'(z) = nz^{n-1}$ for all $n \in \mathbb{N}$.

  Hence polynomials are differentiable.

- $f(z) = \frac{1}{z}$ is differentiable on $\mathbb{C} \setminus \{0\}$ with $f'(z) = -\frac{1}{z^2}$. By induction, we can show that $f(z) = z^{-n}$ is differentiable with $f'(z) = -nz^{-n-1}$ for all $n \in \mathbb{N}$.

  Hence rational functions are differentiable on their domains.

Lecture 11 · 2026-02-17

Proposition 3.5 (Chain Rule)

Let $U, V \in \mathbb{C}$ and $f : U \to V$, $g : V \to \mathbb{C}$. Suppose that f is differentiable at $a \in U$ and g is differentiable at $f(a) \in V$. Then $g \circ f : U \to \mathbb{C}$ is differentiable at $a \in U$, and we have

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

It will be convenient to have an alternative characterization of derivative to prove this. It is common to see $f'(a)$ as the slope of the tangent line to the graph of f at a.

Lemma 3.6

Let $f : X \subseteq \mathbb{C} \to \mathbb{C}$. Then f is differentiable at $a \in X$ iff $\exists A \in \mathbb{C}$ and function $\varepsilon : \{z : z + a \in X\} \to \mathbb{C}$ satisfying $\varepsilon(h) \to 0$ as $h \to 0$ such that

$$f(a+h) = f(a) + Ah + \varepsilon(h)|h|.$$

Remark. This means $f(a+h) \approx f(a) + Ah$ for small h, and the function $\varepsilon(h)$ is to quantify the error of this approximation. We can equivalently write

$$f(a+h) = f(a) + Ah + o(|h|).$$

Moreover, if f is differentiable at a, then we must have $A = f'(a)$.

Proof.

[$\Leftarrow$] We have

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{Ah + \varepsilon(h)|h|}{h} \\ &= A + \lim_{h \to 0}\left(\varepsilon(h)\frac{|h|}{h}\right) \\ &= A \end{aligned}$$

since $\varepsilon(h) \to 0$ as $h \to 0$, and $\frac{|h|}{h}$ is bounded. Hence f is differentiable at a with $f'(a) = A$.

[$\Rightarrow$] Choose $A = f'(a)$, so that $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = A$, or equivalently,

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Ah}{h} = 0.$$

Take

$$\varepsilon(h) = \begin{cases} \frac{f(a+h)-f(a)-Ah}{|h|} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

Then $\varepsilon(h) \to 0$ as $h \to 0$, and the required equality holds.

Proof. [of Chain Rule 3.5]

Since f, g are differentiable at $a, f(a)$ respectively, there exists error functions $\varepsilon_f, \varepsilon_g$ such that

$$\begin{cases}\varepsilon_f(h) \to 0 & \text{as } h \to 0\\ \varepsilon_g(k) \to 0 & \text{as } k \to 0\end{cases}$$

and

$$\begin{aligned} f(a+h) &= f(a) + f'(a)h + \varepsilon_f(h)|h|,\\ g(f(a)+k) &= g(f(a)) + g'(f(a))k + \varepsilon_g(k)|k|. \end{aligned}$$

So we have

$$\begin{aligned}(g\circ f)(a+h) - (g\circ f)(a) &= g(f(a+h)) - g(f(a))\\ &= g\left(f(a) + \underbrace{f'(a)h + \varepsilon_f(h)|h|}_{k}\right) - g(f(a))\\ &= \cancel{g(f(a))} + g'(f(a))\big(f'(a)h + \varepsilon_f(h)|h|\big) + \varepsilon_g(k)|k| - \cancel{g(f(a))}\\ &= g'(f(a))f'(a)h + \underbrace{g'(f(a))\varepsilon_f(h)|h| + \varepsilon_g(k)|k|}_{\varepsilon(h)|h|}.\end{aligned}$$

So, we just need to show that $\varepsilon(h) \to 0$ as $h \to 0$. We have

$$\varepsilon(h) = g'(f(a))\varepsilon_f(h) + \varepsilon_g\big(hf'(a) + \varepsilon_f(h)|h|\big)\big|f'(a) + \varepsilon_f(h)\big|$$

and since

$$\begin{aligned}\lim_{h\to 0}\big[g'(f(a))\varepsilon_f(h)\big] &= g'(f(a))\lim_{h\to 0}\varepsilon_f(h) = 0\\ \lim_{h\to 0}\Big[\varepsilon_g\big(hf'(a) + \varepsilon_f(h)|h|\big)\big|f'(a) + \varepsilon_f(h)\big|\Big] &= \lim_{k\to 0}\varepsilon_g(k)\big|g'(f(a)) + \varepsilon_f(h)\big| = 0,\end{aligned}$$

our conclusion follows.

Example 3.7

Consider the function

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0\\ 0 & x = 0\end{cases}$$

At $x \neq 0$, we have

$$f'(x) = \sin\left(\frac{1}{x}\right) + x\cos\left(\frac{1}{x}\right)\frac{-1}{x^2} = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right).$$

At $x = 0$,

$$f'(0) = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\sin\left(\frac{1}{h}\right),$$

which does not exist. Hence f is only differentiable on $\mathbb{R}\setminus\{0\}$.

Lemma 3.8

If $X \subseteq \mathbb{C} \to \mathbb{C}$ is differentiable at $a \in X$, then f is continuous at a.

Proof. We have

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0}(f(a) + f'(a)h + \varepsilon(h)|h|) = f(a).$$

3.2 Mean Value Theorems

We have concluded that the derivative of a function is the instantaneous rate of change. We want to relate this to the average rate of change.

Theorem 3.9 (Mean Value Theorem)

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then $\exists c \in (a,b)$ such that

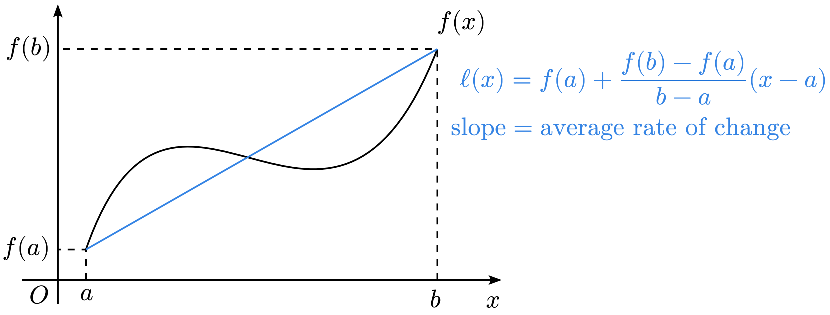
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We shall consider an easier case first.

Proposition 3.10 (Rolle's Theorem)

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then $\exists c \in (a,b)$ such that $f'(c) = 0$.

Proof. [of Mean Value Theorem 3.9]



Let $\varphi(x) = f(x) - \ell(x)$. Then $\varphi(a) = \varphi(b) = 0$ but also $\varphi'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.

By Rolle's Theorem 3.10, there exists $c \in (a,b)$ such that $\varphi'(c) = 0$, so we have $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Lecture 12 · 2026-02-19

Remark. Mean Value Theorem 3.9 can be rephrased as the followings.

Given h such that $a + h \in [a,b]$, $\exists \theta = \theta(h) \in (0,1)$ such that $f(a+h) = f(a) + hf'(a + \theta h)$.

Note that there is no error function ε in this statement.

Proof. [of Rolle's Theorem 3.10]

The idea is that we are looking for some c that is a local minimum or a local maximum.

By Extreme Value Theorem 2.21, f attains its maximum and its minimum on $[a,b]$, *i.e.* $\exists x_m, x_M \in [a,b]$ such that

$$\sup_{[a,b]} f = f(x_M), \quad \inf_{[a,b]} f = f(x_m).$$

If we can show one of x_m, x_M is in (a,b), then by the we have our result:

- $x_m \in (a,b) \Rightarrow \operatorname{sign}\left(\frac{f(x_m+h)-f(x_m)}{h}\right) = \operatorname{sign}(h)$. Hence by taking limits from $h \to 0^+$ and $h \to 0^-$, we have $f'(x_m) = 0$.
- $x_M \in (a,b) \Rightarrow \operatorname{sign}\left(\frac{f(x_M+h)-f(x_M)}{h}\right) = -\operatorname{sign}(h)$. Hence by taking limits from $h \to 0^+$ and $h \to 0^-$, we have $f'(x_M) = 0$.

Now, suppose f is not constant (or otherwise the result is trivial). Either $f(a) < f(b)$ or $f(a) > f(b)$, so either x_M or x_m is in (a,b).

We shall see some applications about Mean Value Theorem 3.9.

Corollary 3.11

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then

1. if $f' \geqslant 0$ on (a,b), then f is increasing on $[a,b]$;
2. if $f' \leqslant 0$ on (a,b), then f is decreasing on $[a,b]$;
3. if $f' = 0$ on (a,b), then f is constant on $[a,b]$.

The monotonocity of f is strict if the inequalities are strict.

Remark. It is not always possible to replace $[a,b]$ with some sets $X \subseteq \mathbb{R}$. For example, consider the function $f : \mathbb{Q} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x^2 > 2 \\ 1 & x^2 < 2 \end{cases}$$

Note that by definition, f is continuous and differentiable at every point in $\mathbb{Q}$, and $f' = 0$ on $\mathbb{Q}$. However, f is not constant on $\mathbb{Q}$.

Note that we can generalise Corollary of Mean Value Theorem 3.11 (3) to functions defined on $\mathbb{C}$.

Lemma 3.12

Let $f : \mathbb{C} \to \mathbb{C}$ be differentiable in $\mathbb{C}$ and $f'(z) = 0$ for all $z \in \mathbb{C}$. Then f is constant on $\mathbb{C}$.

Proof. The idea is to reduce to the $\mathbb{R}$ case. Fix some $z \in \mathbb{C}$, take $g(t) = f(tz)$ where $g : [0,1] \to \mathbb{C}$. Note that g is continuous on $[0,1]$ and differentiable on $(0,1)$, so we can apply Mean Value Theorem 3.9 and its corollaries to $\operatorname{Re} g(t)$ and $\operatorname{Im} g(t)$ separately. We have

$$g'(t) = zf'(tz) = 0 \Rightarrow \begin{cases} \operatorname{Re} g'(t) = 0 \\ \operatorname{Im} g'(t) = 0 \end{cases}$$

Hence $g(t)$ is constant on $[0,1]$, so we have $f(z) = g(1) = g(0) = f(0)$.

Since z is arbitrary, we have $f(z) = f(0)$ for all $z \in \mathbb{C}$, so f is constant on $\mathbb{C}$.

Theorem 3.13 (Inverse Function Theorem, Version 2)

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Assume that $f'(x) > 0$ for all $x \in (a,b)$. Then $f : [a,b] \to [f(a), f(b)]$ is bijective with inverse $f^{-1} : [f(a), f(b)] \to [a,b]$ continuous and differentiable on $(f(a), f(b))$ with

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))} \quad \forall y \in (f(a), f(b)).$$

Proof. By the corollary above, f is strictly increasing, so Inverse Function Theorem (Version 1) 2.25 applies, so f is a bijection to its image and f^{-1} is continuous.

Now for differentiability, let $y \in (f(a), f(b))$ and $x = f^{-1}(y)$. This x is unique by bijectivity. Given h such that $y + h \in (f(a), f(b))$, define k such that $y + h = f(x+k) \Leftrightarrow k = f^{-1}(t+h) - x$. Then

$$\frac{f^{-1}(y+h) - f^{-1}(y)}{h} = \frac{x + k - x}{f(x+k) - y} = \frac{k}{f(x+k) - f(x)} = \frac{1}{\frac{f(x+k) - f(x)}{k}}.$$

So differentiability and the formula holds if we show that $h \to 0$ implies $k \to 0$. This is true because

$$\lim_{h \to 0} k = \lim_{h \to 0} \left(f^{-1}(y+h) - f^{-1}(y)\right) = 0$$

by continuity of f^{-1}.

Example 3.14

- Fix $R > 0$. Define $f : [0,R] \to \mathbb{R}$ with $x \mapsto x^N$ where $N \in \mathbb{N}$. Then f is continuous on $[0,R]$ and differentiable on $(0,R)$ with $f'(x) = Nx^{N-1} > 0$ for all $x \in (0,R)$. Thus by Inverse Function Theorem (Version 2) 3.13, f is a bijection to its image $\left[0, R^N\right]$ with inverse $f^{-1} : \left[0, R^N\right] \to [0,R]$ continuous and differentiable on $\left(0, R^N\right)$ with

  $$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{N(f^{-1}(y))^{N-1}}.$$

  We can write $f^{-1}(y) = y^{\frac{1}{N}}$, so we have $\left(f^{-1}\right)'(y) = \frac{1}{Ny^{\frac{N-1}{N}}} = \frac{1}{N} y^{\frac{1}{N}-1}$.

- $f(x) = e^x$ is continuous and differentiable on $\mathbb{R}$ with $f'(x) = e^x > 0$. Then there exists an inverse

  $$f^{(-1)'}(y) = \frac{1}{e^{f^{-1}(y)}} = \frac{1}{y}.$$

Proposition 3.15 (Cauchy Mean Value Theorem)

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists c \in (a, b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

Proof. We aim to reduce to Rolle's Theorem 3.10. Let

$$\begin{aligned} \varphi(x) &= [g(x) - g(a)][f(b) - f(a)] - [g(b) - g(a)][f(x) - f(a)] \\ \varphi'(x) &= g'(x)[f(b) - f(a)] - f'(x)[g(b) - g(a)], \end{aligned}$$

and note that $\varphi(a) = 0 = \varphi(b)$, Hence by Rolle's Theorem 3.10, there exists $c \in (a, b)$ such that $\varphi'(c) = 0$, so the result follows.

Proposition 3.16 (L'Hôpital's Rule)

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Assume that $g'(x) \neq 0$ for all $x \in (a, b)$ and that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ also exists and is equal to $\lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Example 3.17

We have seen that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. With L'Hôpital's rule, we can also compute this limit as follows:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Behind the scenes in this computation, we are saying that $\exists \theta \in (0, 1)$ such that

$$\begin{aligned} \frac{\sin x}{x} &= \frac{\sin x - \sin 0}{x - 0} = \frac{\cos(\theta x)}{1} && \text{by Cauchy MVT} \\ \lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \cos \theta x = 1 && \text{by continuity of cosine.} \end{aligned}$$

Lecture 13 · 2026-02-21

Corollary 3.18

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If there exists $x_0 \in (a, b)$ such that $f'(x_0) \neq 0$, then locally around x_0, there is an inverse function for f which is differentiable. [Refer to Example Sheet 3 for further details.]

3.3 Higher Derivatives and Taylor's Theorem

Definition 3.19 (Higher Derivatives)

Let $f : X \subseteq \mathbb{C} \to \mathbb{C}$ be differentiable on X. We say that f is **twice differentiable** if

$$\begin{aligned} f' : X &\to R \\ x &\mapsto f'(x) \end{aligned}$$

is differentiable. We similarly define **thrice differentiable** and ***n* times differentiable** for $n \in \mathbb{N}$ inductively.

We say that f is k-times continuously differentiable, and write $f \in C^k(X)$, if f is k times differentiable and

$$\begin{aligned} f^k : X &\to \mathbb{R} \\ x &\mapsto f^k(x) \end{aligned}$$

is continuous.

Remark. Recall that f being differentiable implies its continuity, so if f is k-times differentiable, then f^i is continuous for all $i < k$.

Definition 3.20 (Smoothness)

We say $f : X \subseteq \mathbb{C} \to \mathbb{C}$ is smooth if f is k-times differentiable for all $k \in \mathbb{N}$. We write $f \in C^\infty(X)$.

We saw that f being differentiable implies that it can be well-approximated by a linear function near a:

$$f(x) \approx f(a) + f'(a)(x - a).$$

Meanwhile, if f is twice differentiable, then we can do better:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

We want to state a general version of this apprixmation, under appropriate conditions, and also quantify the error of this approximation.

Definition 3.21 (Taylor Polynomial)

If $f : X \to \mathbb{C}$ is n-times differentiable, $x_0 \in X$, then we call

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

the **Taylor polynomial** of f at x_0 of degree n. We denote this by $T_{n,f,x_0}(x)$.

If f is smooth, we call

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

the **Taylor series** of f at x_0.

Theorem 3.22 (Taylor's Theorem: Lagrange Remainder)

Let $f : [a, a+b]$ be continuous, and assume its first $n-1$ derivatives are continuous as well, and that it is n-times differentiable on $(a, a+h)$. [This is all satisfied if $f \in C^n((c,d))$ where $[a, a+h] \subseteq (c,d)$.]

Let the Taylor remainder be

$$\begin{aligned} R_{n,f,a}(h) &= f(a+h) - T_{n-1,f,a}(h) \\ &= f(a+h) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} h^k. \end{aligned}$$

Then there exists $\theta \in (0,1)$ such that

$$R_{n,f,a}(h) = \frac{h^n}{n!} f^{(n)}(a+\theta h).$$

Remark. For $n = 1$, this is just Mean Value Theorem 3.9. There is nothing special about $h > 0$; if $h = -t < 0$ and f and its $n-1$ derivatives are continuous on $[a-t, a]$ and its n-th derivative exists on $(a-t, a)$, then we can apply the above to $g(x) = f(-x)$:

$$\begin{aligned} g(-a+t) &= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} f^k(a) t^k + (-1)^n \frac{t^n}{n!} f^{(n)}(a - \theta t) \\ f(a+h) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (-t)^k + \frac{(-t)^n}{n!} f^{(n)}(a+\theta h). \end{aligned}$$

Remark. If $f \in C^n((c,d))$ with $[a, a+b] \subseteq (c,d)$, then $f^{(n)}$ is continuous and hence bounded on $[a, a+h]$. Hence $\exists M_n$ such that

$$M_n = \sup_{x \in [a,a+h]} \left| f^{(n)}(x) \right|.$$

Hence

$$\left| R_{n,f,a}(h) \right| = \left| \frac{h^n}{n!} f^{(n)}(a+\theta h) \right| \leqslant M_n \frac{h^n}{n!}.$$

Therefore $R_{n,f,a}(h)$ is $O(h^n)$ as $h \to 0$. Note that this does not tell use that $R_{n,f,a}(h) \to 0$ as $n \to \infty$, since even if $f \in C^\infty$, we do not know how M_n behaves with n.

Proof.

Proof 1 WLOG take $a = 0$. [If $a \neq 0$, apply to $g(x) = f(x+a)$ instead.] Let φ be a continuous function, with its first $n-1$ derivatives continuous and the n-th derivative existing on $(0, h)$, defined by

$$\begin{aligned} \varphi : [0, h] &\to \mathbb{R} \\ t &\mapsto f(t) - T_{n-1,f,0}(t) - \frac{t^n}{n!} B \end{aligned}$$

where we pick B such that $\varphi(h) = 0$. Note that $\varphi(0) = 0$ and also $\varphi^k(0) = 0$ for all $k < n$. By Rolle's theorem,

$$\varphi(0) = \varphi(h) = 0 \Rightarrow \exists \theta_1 \in (0,1) \text{ s.t. } \varphi'(\theta_1 h) = 0.$$
$$\varphi'(0) = \varphi'(\theta_1 h = 0 \Rightarrow \exists \theta_2 \in (0,1) \text{ s.t. } \varphi''(\theta_2\theta_1 h) = 0.$$
$$\vdots$$
$$\varphi^{(n-1)}(0) = \varphi^{(n-1)}(\theta_{n-1} ... \theta_2\theta_1 h) = 0 \Rightarrow \exists \theta_n \in (0,1) \text{ s.t. } \varphi^{(n)}(\theta_n ... \theta_2\theta_1 h) = 0.$$

Now let $\theta = \theta_n \cdot ... \cdot \theta_1$. Then $\theta \in (0,1)$ and we have effective shown that

$$\exists \theta(0,1) \text{ s.t. } f^{(n)}(\theta h) - B = \varphi^{(n)}(\theta h) = 0$$

and hence $B = f^{(n)}(\theta h)$, so we have

$$R_{n,f,0}(h) = f(h) - T_{n-1,f,0}(h) = \frac{h^n}{n!} f^{(n)}(\theta h).$$

Proof 2 WLOG take $a = 0$. Let

$$g : [0,h] \to \mathbb{R}$$
$$t \mapsto f(h) - \sum_{k=0}^{n-1} \frac{f^k(t)}{k!}(h-t)^k.$$

[Note $g(0) = f(h) - T_{n-1,f,0}(h)$.]

Note that g is continuous on $[a,h]$, differentiable on (a,b) with

$$g'(t) = -\frac{f^{(n)}(t)}{(n-1)!}(h-t)^{n-1}$$

For $p = 1, ..., n$, set

$$\varphi_p(t) = g(t) - \frac{(h-t)^p}{h^p} g(0).$$

Then

$$\varphi_p(h) = \varphi_p(0) = 0 \Rightarrow \exists \theta \in (0,1) \quad \text{s.t.} \quad g'(\theta h) + \frac{p(1-\theta)^{p-1}}{h} g(0) = \varphi_p'(\theta h) = 0$$

i.e. $\exists \theta \in (0,1)$ such that

$$-\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta h) + p\frac{(1-\theta)^{p-1}}{h} g(0) = 0$$
$$\Leftrightarrow f(h) - T_{n-1,f,0}(h) = \frac{h^n}{p(n-1)!}(1-\theta)^{n-p} f^{(n)}(\theta h).$$

Now, we can choose $p = n$ to get the required result.

The second proof leads to an alternative version of Taylor's Theorem.

Theorem 3.23 (Taylor's Theorem: Cauchy Remainder)

Let f as in Taylor's Theorem: Lagrange Remainder 3.22 and define $R_{n,f,a}$ similarly. Then there exists $\theta \in (0,1)$ such that

$$R_{n,f,a}(h) = \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h).$$

Lecture 14 · 2026-02-24

Remark. If $f \in C^n([c,d])$ with $[a,a+h] \subseteq (c,d)$, then as in the previous remark, we can set

$$M_n = \sup_{x\in[a,a+h]} |f^n(x)|$$

Then by Taylor's Theorem: Cauchy remainder 3.23, we have

$$\left|R_{n,f,a}(h)\right| \leqslant M_n \frac{|h|^n}{(n-1)!}$$

It appears that this is not better than the bound in Taylor's Theorem: Lagrange Remainder 3.22. However, the following example shows that the Cauchy remainder can be better than the Lagrange remainder.

Example 3.24

Consider $f(x) = x^q$ where $q \in \mathbb{Q}$, which is smooth on $(0,\infty)$. Clearly

$$f^k(x) = q(q-1)\ldots(q-k+1)x^{q-k}.$$

> ***Remark.*** If $q \in \mathbb{Z}$, then f is exactly its Taylor polynomial of some degree.

Then

$$\begin{aligned} T_{n-1,f,1}(x) &= \sum_{k=0}^{n-1} \underbrace{\frac{q(q-1)\ldots(q-k+1)}{k!}}_{\binom{q}{k}} x^k \\ &= \sum_{k=0}^{n-1} \binom{q}{k} x^k. \end{aligned}$$

We now look at the remainder

$$R_{n,f,1}(x) = (1+x)^q - \sum_{k=0}^{n-1} \binom{q}{k} x^k \quad \text{in } (-1,1).$$

There are two ways to estimate this remainder:

- Lagrange remainder: $\exists\theta \in (0,1)$ such that

$$R_{n,f,1}(x) = \binom{q}{n}(1+\theta x)^{q-n} x^n.$$

  If $n \geqslant q$, then $(1+\theta x)^{q-n} \leqslant 1$ for $x \in (0,1)$, so for any $x \in (0,1)$, we have

$$|R_{n,f,1}(x)| \leqslant \left|\binom{q}{n}x\right|^n \to 0 \quad \text{as} \quad n \to \infty.$$

Note that argument fails for $x \in (-1, 0)$.

- Cauchy remainder: $\exists \theta \in (0,1)$ such that

$$\begin{aligned} R_{n,f,1}(x) &= (1-\theta)^{n-1} n \binom{q}{n} (1+\theta x)^{q-n} x^n \\ &= (1-\theta)^{n-1} q \binom{q-1}{n-1} (1+\theta x)^{q-n} x^n \\ &= q \binom{q-1}{n-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{q-1} x^n. \end{aligned}$$

Now $(1+\theta x) = (1-\theta) + \theta(x+1) \geqslant 1-\theta$, so $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \leqslant 1$ as long as $n \geqslant 1$. Hence

$$\begin{aligned} |R_{n,f,1}(x)| &\leqslant |qx| \left|\binom{q-1}{n-1} x^{n-1}\right| \underbrace{|1+\theta x|^{q-1}}_{\text{no dependence on } n} \\ &\leqslant C \left|\binom{q-1}{n-1} x^{n-1}\right| && \text{for some constant } C > 0 \\ &\to 0 && \text{as } n \to \infty \text{ for every } x \in (-1,1). \end{aligned}$$

Hence, we can see that the Cauchy remainder estimate gives us that the remainder is exponentially small as $n \to \infty$ for all $x \in (-1, 1)$, while the Lagrange could only guarantee it for $x \in (0, 1)$.

Definition 3.25 (Analytic Functions)

We say $f \in C^\infty((c,d))$ if for every $a \in (c,d)$, there exists $r > 0$ such that $\forall |h| < r$, we have

$$f(a+h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} h^k = \lim_{n\to\infty} T_{n,f,a}(h),$$

which is equivalent to saying that

$$\lim_{n\to\infty} R_{n,f,a}(h) = 0.$$

Example 3.26

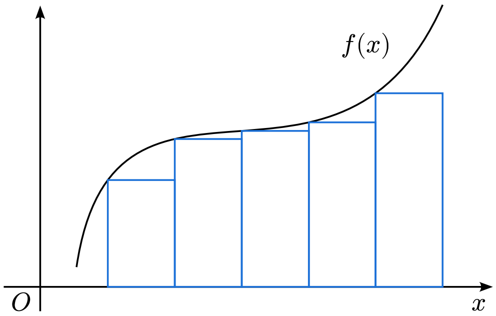
x^q is an analytic function in $(0,2)$, *i.e.* $(1+x)^q$ is analytic in $(-1,1)$, such that $R_{n,f,a}(h) \to 0$ for every $|h| < 1 = r$.

Remark. If $R_{n,f,a}(h) \nrightarrow 0$ as $n \to \infty$, it does not necessarily mean that the Taylor series of f at a does not converge. It instead implies that the Taylor series does not represent f near a.

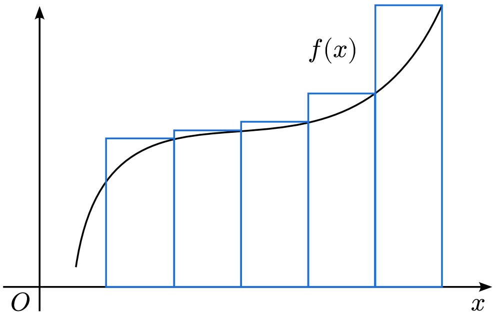
4 Integration

4.1 Basics

We want to define $\int_a^b f(x)\,dx$ as the signed area under the graph of f. If the region enclosed by the graph if a simple shape, then we can define the area using geometry. However, in general, the region may be very complicated, and we need to try to approximate.



Pessimist's Approximation
[Underestimate]



Optimist's Approximation
[Overestimate]

Definition 4.1 (Upper/Lower Riemann Sums)

Let $f : [a, b] \to \mathbb{R}$ be bounded. Given a partition, $\mathcal{P} = \{x_0, x..., x_n\}$ with $a = x_0 < x_1 < ... < x_n = b$, we set

$$L(f, \mathcal{P}) = \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) \inf_{x \in [x_{j-1}, x_j]} f(x)$$

$$U(f, \mathcal{P}) = \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) \sup_{x \in [x_{j-1}, x_j]} f(x)$$

where L is the **lower Riemann sum** and U is the **upper Riemann sum** of f associated with $\mathcal{P}$.

Example 4.2

Consider, on $[0, 1]$, the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

No matter what $\mathcal{P}$ we take, we always have rationals and irrationals in each subinterval $I_j = \left[x_{j-1}, x_j\right]$, hence

$$U(f, \mathcal{P}) = 1 \cdot \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) = 1$$

$$L(f, \mathcal{P}) = 0 \cdot \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) = 0.$$

Remark. If f is bounded, then $\sup_{[a,b]}|f| = k$ for some k, and

$$U(f,\mathcal{P}) \leqslant \sum_{j=1}^{n} \sup_{[a,b]}|f| \left(x_j - x_{j-1}\right) = k(b-a)$$

$$L(f,\mathcal{P}) \geqslant \sum_{j=1}^{n} \left(-\sup_{[a,b]}|f|\right)\left(x_j - x_{j-1}\right) = -k(b-a).$$

Also, $U(f,\mathcal{P}) \geqslant L(f,\mathcal{P})$, hence both

$$\{L(f,\mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\} \quad \text{and} \quad \{U(f,\mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\}$$

are non-empty bounded sets.

Definition 4.3 (Upper/Lower Integral)

Let $f : [a,b] \to \mathbb{R}$ be bounded. We say

$$I_*(f) = \sup_{\mathcal{P}} L(f,\mathcal{P}) \quad \text{and} \quad I^*(f) = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

to be the **lower integral** and **upper integral** of f on $[a,b]$ respectively.

We say f is **Riemann integrable** on $[a,b]$ if $I_*(f) = I^*(f)$, and in this case we set

$$I(f) = \int_a^b f(x)\,\mathrm{d}x = I_*(f) = I^*(f).$$

Remark. By the way we set up the sums, $\int_a^b f\,\mathrm{d}x = -\int_b^a f\,\mathrm{d}x$.

Example 4.4

Take the function on $[0,1]$, defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

as before, we have $I^*(f) = 1$, $I_*(f) = 0$, and f is not Riemann integrable.

Lecture 15 · 2026-02-26

The coarsest partition of $[a,b]$ is $\mathcal{P} = \{a,b\}$. Our intuition is that a finer partition leads to a better estimate for the area.

Lemma 4.5

Let $\mathcal{P},\mathcal{P}'$ be partitions of $[a,b]$, such that $\mathcal{P}' \supseteq \mathcal{P}$, such that $\mathcal{P}'$ is a refinement of $\mathcal{P}$. Then

$$L(f,\mathcal{P}) \leqslant L(f,\mathcal{P}') \leqslant U(f,\mathcal{P}') \leqslant U(f,\mathcal{P}).$$

Proof. We can do this by induction. Let

$$\mathcal{P} = \{x_0, x_1, \ldots, x_n\},$$

and choose some $y \in (x_{k-1}, x_k)$ for some k. Let $\mathcal{P}' = \{y\} \cup \mathcal{P}$. Then

$$\begin{gathered}
\sup_{[x_{k-1},y]} f(x), \sup_{[y,x_k]} f(x) \leqslant \sup_{[x_{k-1},x_k]} f(x) \\
(y - x_{k-1}) \sup_{[x_{k-1},y]} f(x) + (x_k - y) \sup_{[y,x_k]} f(x) \leqslant ((y - x_{k-1}) + (x_k - y)) \sup_{[x_{k-1},x_k]} f(x) \\
(y - x_{k-1}) \sup_{[x_{k-1},y]} f(x) + (x_k - y) \sup_{[y,x_k]} f(x) \leqslant (x_k - x_{k-1}) \sup_{[x_{k-1},x_k]} f(x) \\
U(f, \mathcal{P}') \leqslant U(f, \mathcal{P}).
\end{gathered}$$

So similarly, using inf instead of sup, we can show $L(f, \mathcal{P}) \leqslant L(f, \mathcal{P}')$.

This argument can be repeated for any refinement,

$$\mathcal{P}' \setminus \mathcal{P} = \{y_1, y_2, \ldots, y_m\},$$

and the result follows by induction.

Lemma 4.6

Let $\mathcal{P}, \mathcal{P}'$ be partitions of $[a, b]$ [not necessarily refinements of each other]. Then

$$L(f, \mathcal{P}) \leqslant U(f, \mathcal{P}').$$

Proof. Note that $\mathcal{P}, \mathcal{P}' \subseteq \mathcal{P} \cup \mathcal{P}'$. By Lemma 4.5, we have

$$L(f, \mathcal{P}) \leqslant L(f, \mathcal{P} \cup \mathcal{P}') \leqslant U(f, \mathcal{P} \cup \mathcal{P}') \leqslant U(f, \mathcal{P}').$$

Lemma 4.7

If $f : [a, b] \to \mathbb{R}$, then $I_*(f) \leqslant I^*(f)$.

Proof. We have

$$\begin{gathered}
L(f, \mathcal{P}) \leqslant \inf_{\mathcal{P}'} U(f, \mathcal{P}') \\
\sup_{\mathcal{P}} L(f, \mathcal{P}) \leqslant \inf_{\mathcal{P}'} U(f, \mathcal{P}') \\
I_*(f) \leqslant I^*(f).
\end{gathered}$$

Remark. This means that f is integrable iff $I_*(f) \geqslant I^*(f)$.

4.2 Integrability Criteria

There is another way to define integrals. No matter how small the threshold is, we can always find a partition of $[a, b]$ so that the gap between the optimistic and pessimistic estimates is smaller than the threshold. Our definition of integrability is equivalent to this.

Proposition 4.8 (Riemann Integrability Criteria)

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable iff

$$\forall \varepsilon > 0, \exists \mathcal{P} = \mathcal{P}(\varepsilon) \text{ partition of } [a,b] : U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon.$$

Proof.

[$\Leftarrow$] We have $\forall \varepsilon > 0$,

$$0 \leqslant I^*(f) - I_*(f) \leqslant U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon.$$

Hence $I^*(f) = I_*(f)$.

[$\Rightarrow$] From the definition of sup and inf, we can always find partitions $\mathcal{P}, \mathcal{P}'$ such that

$$U(f,\mathcal{P}) \leqslant I^*(f) + \frac{\varepsilon}{2}$$
$$L(f,\mathcal{P}') \geqslant I_*(f) - \frac{\varepsilon}{2}$$

By assumption, $I^*(f) = I_*(f)$, so $U(f,\mathcal{P}) - L(f,\mathcal{P}') \leqslant \varepsilon$. To conclude, we can take $\mathcal{P}'' = \mathcal{P} \cup \mathcal{P}'$, and by Lemma 4.5, we have

$$U(f,\mathcal{P}'') - L(f,\mathcal{P}'') \leqslant U(f,\mathcal{P}) - L(f,\mathcal{P}') \leqslant \varepsilon < 2\varepsilon.$$

It can be shown that this is equivalent to a sequential version.

Proposition 4.9

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable iff there exists a sequence of partitions $(\mathcal{P}_n)$ of $[a,b]$ such that

$$U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) \to 0 \quad \text{as } n \to \infty.$$

Exercise. The proof is left as an exercise.

Example 4.10

1. Consider $f(x) = x$ on $[0,1]$. Let $\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Then

$$U(f,\mathcal{P}_n) = \sum_{j=1}^{n} \left(\sup_{[\frac{j-1}{n}, \frac{j}{n}]} x \right) \frac{1}{n} = \frac{1}{n^2} \sum_{j=1}^{n} \sup_{[j-1,j]} x = \frac{1}{n^2} \sum_{j=1}^{n} j = \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

$$L(f,\mathcal{P}_n) = \sum_{j=1}^{n} \left(\inf_{[\frac{j-1}{n}, \frac{j}{n}]} x \right) \frac{1}{n} = \frac{1}{n^2} \sum_{j=1}^{n} \inf_{[j-1,j]} x = \frac{1}{n^2} \sum_{j=1}^{n} (j-1) = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$

   - With original definition,

$$I_*(f) = \sup_{\mathcal{P}} L(f,\mathcal{P}) \geqslant \sup_n L(f,\mathcal{P}_n) = \frac{1}{2}$$
$$I^*(f) = \inf_{\mathcal{P}} U(f,\mathcal{P}) \leqslant \inf_n U(f,\mathcal{P}_n) = \frac{1}{2}$$

So $I_*(f) \geqslant \frac{1}{2} \geqslant I^*(f)$. By $I^*(f) \geqslant I_*(f)$, we have $\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{2}$, and the function is integrable.

- With Riemann Integrability Criteria 4.8, we have

$$U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

So f is integrable.

2. Consider on $[0,1]$ the function

$$f(x) = \begin{cases} 1 & \text{if } x \leqslant \frac{1}{2} \\ 0 & \text{if } x > \frac{1}{2} \end{cases}$$

Take $\mathcal{P}_n$ as before for $n \geqslant 3$. Then

$$\sup_{x_{k-1},x_k} f - \inf_{x_{k-1},x_k} f = \begin{cases} 1 & \text{if } \frac{1}{2} \in (x_{k-1}, x_k) \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = 1 \cdot \frac{1}{n} = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

By Proposition 4.9, f is integrable.

We can try to generalize the above examples to get classes of functions that are integrable.

Proposition 4.11

Let $f : [a,b] \to \mathbb{R}$. If f is monotone, then f is Riemann integrable.

Lecture 16 · 2026-02-28

Proof. If f is monotone, then f is bounded by $\max\{|f(a)|, |f(b)|\}$.

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \sum_{j=1}^{n} \left(\sup_{I_j} f - \inf_{I_j} f \right) \left| I_j \right| \quad \text{where } I_j = \left[x_{j-1}, x_j \right].$$

Since f is monotone, $\sup_{I_j}$ and $\inf_{I_j}$ are attained at the endpoints of I_j. WLOG assume f is increasing, then

$$\sup_{I_j} = f\left(x_j\right), \quad \inf_{I_j} = f\left(x_{j-1}\right).$$

Hence, for any partition $\mathcal{P}$ of $[a,b]$,

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \sum_{j=1}^{n} \left(f\left(x_j\right) - f\left(x_{j-1}\right)\right)\left|I_j\right|.$$

Set $\mathcal{P}_n = \left\{a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, \ldots, b\right\}$, so that $\left|I_j\right| = \frac{b-a}{n}$ for all j. Then

$$U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = \frac{b-a}{n}\sum_{j=1}^{n}\left(f\left(x_j\right) - f\left(x_{j-1}\right)\right) = \frac{b-a}{n}(f(b) - f(a)) \to 0 \quad \text{as } n \to \infty.$$

Proposition 4.12 (Continuity Implies Integrability)

If $f : [a,b] \to \mathbb{R}$ is continuous, then it is integrable.

Proof. By Extreme Value Theorem 2.21, f is continuous on $[a,b]$ with $|a|, |b| < \infty$, so f is bounded.

We will show that contrapositive, that if f is not integrable, then f is not continuous.

If f is not integrable, $\exists \varepsilon > 0$ such that $\forall \mathcal{P}$ partition of $[a,b]$,

$$\begin{aligned} \varepsilon < U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{j=1}^{n}\left|I_j\right|\left(\sup_{I_j} f - \inf_{I_j} f\right) \\ &\leqslant \sum_{k=1}^{n}\left|I_k\right| \max_{1\leqslant j\leqslant n}\left(\sup_{I_j} f - \inf_{I_j} f\right) \\ &= (b-a)\max_{1\leqslant j\leqslant n}\left(\sup_{I_j} f - \inf_{I_j} f\right). \end{aligned}$$

Hence $\exists j$ such that $\sup_{I_j} f - \inf_{I_j} f > \frac{\varepsilon}{b-a}$.

By Extreme Value Theorem 2.21, $\exists y, z \in \left[x_{j-1}, x_j\right]$ such that $f(y) - f(z) > \frac{\varepsilon}{b-a}$.

Since the above is true for all partitions, it should also be true for the $\mathcal{P}_n$ in the previous proof. So for each n, we can get y_n, z_n with $\left|y_n - z_n\right| \leqslant \frac{b-a}{n}$ and $\left|f(y_n) - f(z_n)\right| > \frac{\varepsilon}{b-a}$.

From this point, we want to show that f is not sequentially continuous. The problem is that we do not know if $(y_n), (z_n)$ are convergent. However, since (y_n) and (z_n) are both bounded, by Bolzano-Weierstrass Theorem 1.13, we can find subsequences $\left(y_{n_k}\right), \left(z_{n_k}\right)$ that are convergent.

Let $y = \lim_{k\to\infty} y_{n_k}$ and $z = \lim_{k\to\infty} z_{n_k}$. Then $y, z \in [a,b]$, and by

$$\left|z_{n_k} - y_{n_k}\right| < \frac{b-a}{n_k} \to 0 \quad \text{as } k \to \infty,$$

we have $y = z$. We can now invoke the sequential continuity: we have

$$\begin{gathered} y_{n_k}, z_{n_k} \to y = z \quad \text{as } k \to \infty, \\ \left|f\left(y_{n_k}\right) - f\left(z_{n_k}\right)\right| > \frac{\varepsilon}{b-a} \quad \text{as } k \to \infty, \end{gathered}$$

hence $f\left(y_{n_k}\right)$ and $f\left(z_{n_k}\right)$ do not converge to the same limit, and f is not sequentially continuous, hence not continuous.

Proposition 4.13 (Piecewise Continuity Implies Integrability)

If $f : [a, b] \to \mathbb{R}$ is piecewise continuous, *i.e.* suppose there is a partition $\mathcal{P}$ of $[a, b]$ such that $f|_{(x_{j-1},x_j)}$ is continuous and has a finite limit as $x \to x_j^-$ and $x \to x_j^+$, for all $j = 1, \dots, n$, then f is integrable, and

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \overline{f_j}(x)\,\mathrm{d}x$$

where

$$\overline{f_j}(x) = \begin{cases} f(x) & \text{if } x \in \left(x_{j-1}, x_j\right) \\ \lim_{x \to x_j^-} f(x) & \text{if } x = x_j \\ \lim_{x \to x_{j-1}^+} f(x) & \text{if } x = x_{j-1} \end{cases}$$

Proof.

Lemma 4.14

Let $f, g : [a, b] \to \mathbb{R}$ be bounded. Suppose g is integrable, and the set

$$\{x \in [a, b] : f(x) \neq g(x)\} = \{z_1, \dots, z_N\}$$

is finite, then f is integrable and $\int_a^b f = \int_a^b g$.

Proof. Set $M = \sup_{[a,b]} f$. Fix $\varepsilon > 0$ then $\exists \mathcal{P}$ of $[a, b]$ such that

$$U(g, \mathcal{P}) - L(g, \mathcal{P}) < \varepsilon.$$

The idea is to choose a partition which isolates problematic points, and gives them very little weight. Choose intervals $J_j = [z_k - r_k, z_k + r_k]$ with

$$\sum_{n=1}^{N} |J_k| = 2 \sum_{n=1}^{N} r_k < \varepsilon.$$

Set $\mathcal{P}' = \mathcal{P} \cup \{z_1 \pm r_1, z_2 \pm r_2, \dots, z_N \pm r_N\}$. Also, let $J = \bigcup_{n=1}^{N} J_n$. Then

$$U(g, \mathcal{P}') - L(g, \mathcal{P}') \leqslant U(g, \mathcal{P}) - L(g, \mathcal{P}) < \varepsilon.$$

Try to estimate

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum_{j=1}^{n} \left|I_j\right| \left(\sup_{J_j} f - \inf_{J_j} f \right) \\ &= \sum_{j=1, I_j \subseteq J}^{n} \left|I_j\right| \left(\sup_{J_j} f - \inf_{J_j} f \right) + \sum_{j=1, I_j \nsubseteq J}^{n} \left|I_j\right| \left(\sup_{J_j} f - \inf_{J_j} f \right). \end{aligned}$$

Note that if $I_j \nsubseteq J$, then $f = g$, and hence

$$\sum_{j=1, I_j \subseteq J}^{n} \left|I_j\right| \left(\sup_{J_j} f - \inf_{J_j} f\right) \leqslant \sum_{j=1}^{n} \left|I_j\right| \left(\sup_{J_j} g - \inf_{J_j} g\right) = U(g,\mathcal{P}') - L(g,\mathcal{P}') < \varepsilon,$$

and if $I_j \subseteq J$, then we only know that f is bounded, and

$$\left(\sup_{I_j} f - \inf_{I_j} f\right) \leqslant 2M.$$

Hence,

$$\sum_{j=1, I_j \subseteq J}^{n} \left|I_j\right| \left(\sup_{I_j} f - \inf_{I_j} f\right) \leqslant 2M \sum_{j=1, I_j \subseteq J}^{n} \left|I_j\right| \leqslant 2M \sum_{n=1}^{N} |J_n| < 2M\varepsilon.$$

Thus,

$$U(f,\mathcal{P}') - L(f,\mathcal{P}') < (2M+1)\varepsilon.$$

For the formula for the integral, we have

$$L(f,\mathcal{P}') \leqslant \int_a^b f \leqslant U(f,\mathcal{P}'),$$
$$L(g,\mathcal{P}') \leqslant \int_a^b g \leqslant U(g,\mathcal{P}').$$

Hence

$$\int_a^b f - \int_a^b g \leqslant \underbrace{U(f,\mathcal{P}') - U(g,\mathcal{P}')}_{<\varepsilon} + \underbrace{(g,\mathcal{P}') - L(g,\mathcal{P}')}_{<\varepsilon}.$$

Thus $\int_a^b f - \int_a^b g < 2\varepsilon$. Similarly, we can show $\int_a^b g - \int_a^b f < 2\varepsilon$, and the result follows by arbitrariness of ε.

By Lemma 4.14 we have that $f|_{[x_{j-1},x_j]}$ is integrable and $\int_{x_{j-1}}^{x_j} f = \int_{x_{j-1}}^{x_j} \overline{f_j}$. To conclude, we just need to show that

Proposition 4.15

Let $f : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. Then f is integrable iff $f|_{[a,c]}$ and $f|_{[c,b]}$ are integrable. Moreover,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Lecture 17 · 2025-03-03

Proof. We will set $I_L = [a,c]$, $I_R = [c,b]$, $f_L = f|_{I_L}$ and $f_R = f|_{I_R}$ for brevity.

[⇒] For $\varepsilon > 0$, then there exists a partition $\mathcal{P}$ of $[c,b]$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$. WLOG let $c = x_\ell$ for some $\ell \in \{0,\dots,n\}$ [otherwise add c to $\mathcal{P}$, which can only make $U(f,\mathcal{P}) - L(f,\mathcal{P})$

smaller]. Then $\mathcal{P} = \mathcal{P}_L \cup \mathcal{P}_R$ where $\mathcal{P}_L = \left\{\underbrace{x_0}_{a}, \dots, \underbrace{x_\ell}_{c}\right\}$ and $\mathcal{P}_R = \left\{\underbrace{x_\ell}_{c}, \dots, \underbrace{x_n}_{b}\right\}$ are partitions of $[a, c]$ and $[c, b]$ respectively. Furthermore,

$$\begin{cases} U(f, \mathcal{P}) = U(f_L, \mathcal{P}_L) + U(f_R, \mathcal{P}_R) \\ L(f, \mathcal{P}) \ = L(f_L, \mathcal{P}_L) + L(f_R, \mathcal{P}_R) \end{cases}$$

gives

$$\underbrace{U(f_L, \mathcal{P}_L) - L(f_L, \mathcal{P}_L)}_{\geqslant 0} + \underbrace{U(f_R, \mathcal{P}_R) - L(f_R, \mathcal{P}_R)}_{\geqslant 0} < \varepsilon.$$

So f_L and f_R are integrable.

[$\Leftarrow$] For $\varepsilon > 0$, $\exists \mathcal{P}_L, \mathcal{P}_R$ partitions of I_L and I_R respectively such that

$$\begin{cases} U(f_R, \mathcal{P}_R) - L(f_R, \mathcal{P}_R) < \varepsilon \\ U(f_L, \mathcal{P}_L) - L(f_L, \mathcal{P}_L) \ < \varepsilon \end{cases}$$

Hence by the equalities in the first part,

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < 2\varepsilon.$$

Hence f is integrable in $[a, b]$. Furthermore,

$$\begin{aligned} \int_a^b f &\geqslant L(f_L, \mathcal{P}_L) + L(f_R, \mathcal{P}_R) \\ &\geqslant \int_a^c f_L + \int_c^b f_R - 2\varepsilon, \\ \int_a^b f &\leqslant \int_a^c f_L + \int_c^b f_R + 2\varepsilon. \end{aligned}$$

Hence, $\left|\int_a^b f - \left(\int_a^c f_L + \int_c^b f_R\right)\right| < 2\varepsilon$. Since ε is arbitrary, the result follows.

Continuous, piecewise continuous and monotone functions are all integrable, we may wonder if these are all the Riemann integrable functions. The answer is negative.

Example 4.16 (Thomae Function)

Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ with } \gcd(p, q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, $L(f, \mathcal{P}) = 0$ for any partition $\mathcal{P}$ of $[0, 1]$, hence $I_*(f) = 0$. We claim that f is integrable, then $\forall \varepsilon > 0, \exists \mathcal{P}$ such that $U(f, \mathcal{P}) < \varepsilon$.

Pick $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Set

$$\begin{aligned} X_N &= \left\{x \in [0,1] : f(x) \geqslant \frac{1}{N}\right\} \\ &\subseteq \left\{\frac{p}{q} : 1 \leqslant q \leqslant N, 0 \leqslant p \leqslant q\right\} \\ &= \{y_1, \dots, y_M\} \quad \text{for some finite } M. \end{aligned}$$

Define $\mathcal{P}$ such that

1. each y_k is some subinterval of $\mathcal{P}$
2. this subinterval has length $< \frac{\varepsilon}{M}$ [we wish to give little weight to the bad points]

Then,

$$\begin{aligned} U(f, \mathcal{P}) &= \sum_{\substack{I \in \mathcal{P} \\ I \cap X_N \neq \varnothing}} |I| \underbrace{\sup_I f}_{\leqslant 1} + \sum_{\substack{I \in \mathcal{P} \\ I \cap X_N = \varnothing}} |I| \underbrace{\sup_I f}_{\leqslant \frac{1}{N}} \\ &\leqslant \sum_{\substack{I \in \mathcal{P} \\ I \cap X_N \neq \varnothing}} |I| + \sum_{\substack{I \in \mathcal{P} \\ I \cap X_N = \varnothing}} |I| \frac{1}{N} \\ &\leqslant \sum_{\substack{I \in \mathcal{P} \\ I \cap X_N \neq \varnothing}} |I| + \frac{1}{N} \sum_{I \in \mathcal{P}} |I| \\ &\leqslant M \cdot \frac{\varepsilon}{M} + \frac{1}{N} \cdot 1 \\ &< 2\varepsilon. \end{aligned}$$

This gives us a different result compared to the Dirichlet function. Despite both functions have infinitely many discontinuities, the integrability properties are fundamentally different.

Proposition 4.17 (Countable Discontinuities Implies Integrability)

If $f : [a, b] \to \mathbb{R}$, and $D = \{x \in [a, b] : f \text{ is not continuous at } x\}$, then

1. D is finite implies that f is Riemann integrable.
2. D is countable implies that f is Riemann integrable.

Proof.

1. See Example Sheet 3 Q13.
2. The proof is non-examinable.

Remark. Proposition 4.17 (1) is a stronger version of Proposition 4.13, since we do not require the one-sided limits to exist at the discontinuities. For example, consider oscillating functions like $\sin\left(\frac{1}{x}\right)$.

> ***Important.*** If f is not Riemann integrable then D cannot be countable. However, there are functions which have uncountably many discontinuities but are still Riemann integrable, such as the indicator function of the Cantor set.

4.3 Basic Properties of Integrals

Lemma 4.18

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Then

1. $f(x) \leqslant g(x)$ for all $x \in [a, b]$ implies $\int_a^b f \leqslant \int_a^b g$.
2. For $\lambda \in \mathbb{R}$, $\int_a^b \lambda f = \lambda \int_a^b f$.
3. $f + g$ is integrable and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
4. $|f|$ is integrable and $\left|\int_a^b f\right| \leqslant \int_a^b |f|$. [Triangle inequality for integrals]
5. fg is integrable but $\int fg \neq \int f \int g$ in general.

In order to show the lemma, we first need a few other lemmas, including

- an intermediate lemma about upper and lower Riemann sums
- an intermediate lemma on sup and inf of functions on intervals.

> ***Exercise.*** Write out the equivalent lemma for the first one, having seen the second one below.

Lemma 4.19

Let I be a closed and bounded interval, and let $f, g : I \to \mathbb{R}$ be bounded. Then

1. If $f(x) \leqslant g(x)$ for all $x \in I$, then sup and inf preserves this inequality.

2.1. $\sup_I(-f) = -\inf_I f$.

2.2. For $\lambda > 0$ fixed, $\sup_I(\lambda f) = \lambda \sup_I f$.

3. $\sup_I(f + g) \leqslant \sup_I f + \sup_I g$, and $\inf_I(f + g) \geqslant \inf_I f + \inf_I g$.
4. $\sup_I|f| - \inf_I|f| \leqslant \sup_I f - \inf_I f$.
5. $\sup_I f^2 - \inf_I f^2 \leqslant 2 \sup|f|(\sup_I f - \inf_I f)$.

Lecture 18 · 2026-03-05

Proof.

> ***Exercise.*** 1–3 are exercises.

4. If $f \geqslant 0$ on all of I, or if $f \leqslant 0$ on all of I, then the result is immediate.

   If $\inf_I f < 0 < \sup_I f$, then

$$\begin{aligned}\sup_I |f| - \inf_I |f| &\leqslant \sup_I |f| \leqslant \max\left(\sup_I f, \sup_I (-f)\right) \\ &\leqslant \sup_I f + \sup_I(-f) = \sup_I f - \inf_I f.\end{aligned}$$

5. We have

$$\begin{aligned}f^2(x) - f^2(y) &= (f(x) - f(y))(f(x) + f(y)) \\ \sup_I f^2 - \inf_I f^2 &\leqslant \sup_{x \in I} \inf_{y \in I} |[f(x) - f(y)][f(x) + f(y)]| \\ &\leqslant 2 \sup_I |f| \underbrace{\sup_{x \in I} \inf_{y \in I} [f(x) - f(y)]}_{\sup_I f - \inf_I f}.\end{aligned}$$

Proof. [of Lemma 4.18]

> ***Exercise.*** 1–3 are exercises.

4. For any partition $\mathcal{P}$ of $[a, b]$,

$$U(|f|, \mathcal{P}) - L(|f|, \mathcal{P}) \leqslant U(f, \mathcal{P}) - L(f, \mathcal{P}).$$

Hence if f is integrable, then $|f|$ is also integrable.

Since for all $x \in [a, b]$,

$$-|f(x)| < f(x) < |f(x)|,$$

by (1) we have

$$-\int_a^b |f| \leqslant \int_a^b f \leqslant \int_a^b |f|.$$

Hence $\left|\int_a^b f\right| \leqslant \int_a^b |f|$.

5. We have

$$fg = \frac{1}{4}\left[(f + g)^2 - (f - g)^2\right]$$

Hence we only need to show that f being integrable implies that f^2 is integrable.

For any partition $\mathcal{P}$ of $[a, b]$,

$$U\left(f^2, \mathcal{P}\right) - L\left(f^2, \mathcal{P}\right) \leqslant 2 \sup_I |f| \cdot [U(f, \mathcal{P}) - L(f, \mathcal{P})].$$

Hence if f is integrable, then f^2 is also integrable.

4.4 Integration and Differentiation

One have probably heard that integration and differentiation are inverse operations. We will make this precise in this section.

We will think about integral of f in this section as

$$F : [a,b] \to \mathbb{R}, \quad F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

If we want F to be differentiable, it must be continuous:

Proposition 4.20 (Integration Is Continuous)

Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable, ans let $F(x) = \int_a^x f(t)\,\mathrm{d}t$. Then F is continuous in $[a,b]$.

Proof.

$$\begin{aligned} |F(x+h) - F(x)| &= \left| \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t \right| \\ &= \left| \int_x^{x+h} f(t)\,\mathrm{d}t \right| \\ &\leqslant \int_x^{x+h} |f(t)|\,\mathrm{d}t \\ &\leqslant \sup_{[a,b]} |f| \underbrace{\int_x^{x+h} \mathrm{d}t}_{h} \to 0 \quad \text{as } h \to 0. \end{aligned}$$

Theorem 4.21 (Fundamental Theorem of Calculus, Part 1)

If $f : [a,b] \to \mathbb{R}$ is Riemann integrable and continuous at x_0, then $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is differentiable at x_0, with

$$F'(x_0) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(t)\,\mathrm{d}t\right]\bigg|_{x=x_0} = f(x_0).$$

Proof. We will use ε-definition of integrability, *i.e.* $\varepsilon(h) = \frac{F(x_0+h) - F(x_0) - hf(x_0)}{|h|}$ and we want to show that $\varepsilon(h) \to 0$ as $h \to 0$.

Estimating numerator, we have

$$\begin{aligned} |F(x_0+h)-F(x_0)-hf(x_0)| &= \left| \int_{x_0}^{x_0+h} f(t)\,\mathrm{d}t - hf(x_0) \right| \\ &= \left| \int_{x_0}^{x_0+h} |f(t)-f(x_0)|\,\mathrm{d}t \right| \\ &\leqslant \int_{x_0}^{x_0+h} |f(t)-f(x_0)|\,\mathrm{d}t \\ &\leqslant \sup_{t\in[0,h]} |(x_0+t)-f(x_0)| \cdot \underbrace{\int_{x_0}^{x_0+h} \mathrm{d}t}_{h} \\ \Rightarrow |\varepsilon(h)| &\leqslant \sup_{t\in[0,h]} |f(x_0+t)-f(x_0)| \to 0 \quad \text{as } h \to 0. \end{aligned}$$

Example 4.22

Take $[a,b]=[-1,1]$ with

$$f(x)=\begin{cases} -1 & \text{for } x \geqslant 0 \\ 1 & \text{for } x < 0 \end{cases}$$

is integrable, with

$$F(x)=\int_0^x f(t)\,\mathrm{d}t = \begin{cases} -1-x & x \leqslant 0 \\ x-1 & x \geqslant 0 \end{cases} = |x|-1$$

which is not differentiable at $x=0$.

Hence the condition of continuity at x_0 is necessary in Theorem 4.21.

Corollary 4.23

If $f=g'$ is continuous on $[a,b]$, then

$$F(x)=\int_a^x f(t)\,\mathrm{d}t = g(x)-g(a) \quad \forall x \in [a,b].$$

Proof. $(F-g)'=0$ by Theorem 4.21, so by Mean Value Theorem 3.9, $F-g$ is constant, *i.e.*

$$F(x)-g(x)=\underbrace{F(a)}_{0}-g(a) \quad \Leftrightarrow \quad F(x)=g(x)-g(a) \quad \forall x$$

Theorem 4.24 (Fundamental Theorem of Calculus, Part 2)

If $f:[a,b]\to\mathbb{R}$ is Riemann integrable, and there exists differentiable $F:[a,b]\to\mathbb{R}$ such that $F'=f$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b)-F(a).$$

Proof. By assumption, $\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}$ of $[a,b]$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$.

Applying Mean Value Theorem 3.9 to F on intervals of this partition,

$$\begin{aligned} & F\left(x_j\right) - F\left(x_{j-1}\right) = f\left(t_j\right)\left(x_j - x_{j-1}\right) \quad \text{for some } t_j \in \left[x_{j-1}, x_j\right]. \\ \Rightarrow \quad & F(b) - F(a) = \sum_{j=1}^{n} f\left(t_j\right)\left(x_j - x_{j-1}\right) \ni [L(f,\mathcal{P}), U(f,\mathcal{P})] \\ \Rightarrow \quad & I_*(f) \leqslant F(b) - F(a) \leqslant I^*(f). \end{aligned}$$

Since f is integrable, $I^*(f) = I_*(f) = I(f) = \int_a^b f$, hecne $F(b) - F(a) = \int_a^b f$.

Lecture 19 · 2026-03-07

Now we shall derive some common consequences.

Proposition 4.25 (Integration by Parts)

Suppose $f, g \in C^1([a,b])$. Then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

Proof. By Fundamental Theorem of Calculus, Part 2 4.24 and product rule,

$$f'g = (fg)' - fg'$$

Integrate in (a,b) and using FTC, the result follows.

Proposition 4.26 (Integration by Substitution)

Let $f : [a,b] \to \mathbb{R}$ be continuous and let $g \in C^1([\alpha,\beta])$ with $g : [\alpha,\beta] \to [a,b]$, and $g(\alpha) = a$, $g(\beta) = b$. Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t.$$

Proof. Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$, then $F : [a,b] \to \mathbb{R}$ is well-defined and differentiable by Theorem 4.21. Set $h = F \circ g : [\alpha,\beta] \to \mathbb{R}$. Then h is differentiable:

$$h'(t) = F'(g(t))g'(t) = f(g(t))g'(t).$$

Hence,

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x &= F(b) - F(a) = F(g(\beta)) - F(g(\alpha)) = h(\beta) - h(\alpha) \\ &= \int_\alpha^\beta h'(t)\,\mathrm{d}t \\ &= \int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t. \end{aligned}$$

Theorem 4.27 (Taylor's Theorem: Integral Remainder)

Suppose $f \in C^n([a, a+h])$. Let $R_{n,f,a}(h)$ be as before. Then

$$\begin{aligned} R_{n,f,a}(h) &= \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(a+th)\,\mathrm{d}t \\ &= \frac{1}{(n-1)!}\int_0^h (h-u)^{n-1} f^{(n)}(a+u)\,\mathrm{d}u. \end{aligned}$$

Remark. By Extreme Value Theorem 2.21, $\exists M_n = \sup_{x\in[0,n]} \left|f^{(n)}(a+x)\right| < \infty$. Thus, by Taylor's Theorem: Integral Remainder 4.27,

$$|R_{n,f,a}(h)| \leqslant \frac{h^n}{n!}\left|\int_0^1 \cdots\right| = \frac{h^n}{n!}M_n\int_0^1 \mathrm{d}t = \frac{h^n}{n!}.$$

Hence $R_{n,f,a}(h) \to 0$ as $h \to 0$ for fixed n.

If we knew $\sup_{n\geqslant 0} M_n = M < \infty$, then we would get

$$|R_{n,f,a}(h)| \leqslant \frac{Mh^n}{h!} \to 0 \quad \text{as} \quad n \to \infty$$

for all $|h| < 1$, and this would mean f to be analytic at a.

Remark. Taylor's Theorem: Integral Remainder 4.27 generalises to $f : X \subseteq \mathbb{C} \to \mathbb{C}$, where X contains the line segment $[a, a+h]$ (see Example Sheet 3). The reason we don't further explore this is that in $\mathbb{C}$, differentiability implies smoothness, which in turn implies that they are analytic [see IB Complex Analysis]. This comes with estimates on M_n as a function of n, that then one can plug into the above to get convergence of $R_{n,f,a}(h)$ to 0 as $n \to \infty$ for some h.

Proof. Using Integration by Parts 4.25,

$$\begin{aligned} R_{n,f,a}(h) &= \frac{1}{(n-1)!}\int_0^h (h-u)^{n-1} f^{(n)}(a+u)\,\mathrm{d}u \\ &= -h^{n-1}\frac{f^{(n-1)}(a)}{(n-1)!} + \frac{1}{(n-2)!}\int_0^h (h-u)^{n-2} f^{(n-1)}(a+u)\,\mathrm{d}u \\ &= \ldots \\ &= -\sum_{k=1}^{n-1}\frac{f^{(k)}(a)}{k!}h^k + \int_0^h f'(a+u)\,\mathrm{d}u \\ &= f(a+h) - f(a) - \sum_{k=1}^{n-1}\frac{f^{(k)}(a)}{k!}h^k. \end{aligned}$$

We would use Taylor's Theorem: Integral Remainder 4.27 to give an alternative proof of Taylor's Theorem: Cauchy Remainder 3.23 and Taylor's Theorem: Lagrange Remainder 3.22. To do this, we would need to mean value theorem for integrals.

Proposition 4.28 (Cauchy Mean Value Theorem for Integrals)

Let $f, g : [a, b] \to \mathbb{R}$ be continuous, and $g(x) \neq 0$ for all $x \in [a, b]$, then $\exists c \in (a, b)$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(c)\int_a^b g(x)\,\mathrm{d}x.$$

Proof. Apply Cauchy Mean Value Theorem 3.15 to $F(x) = \int_a^x f(t)g(t)\,\mathrm{d}t$ and $G(x) = \int_a^x g(t)\,\mathrm{d}t$, we get $\exists c \in (a, b)$ such that

$$\begin{aligned} (F(b) - F(a))G'(c) &= (G(b) - G(a))F'(c) \\ g(c)\int_a^b f(t)g(t)\,\mathrm{d}t &= f(c)g(c)\int_a^b g(t)\,\mathrm{d}t \\ \int_a^b f(t)g(t)\,\mathrm{d}t &= f(c)\int_a^b g(t)\,\mathrm{d}t. \end{aligned}$$

Proof of TT: Lagrange Remainder 3.22. Assuming continuity of $f^{(n)}$, let $g(t) = (1-t)^{n-1}$, then $\exists \theta \in (0, 1)$ such that

$$\begin{aligned} R_{n,f,a}(h) &= \frac{h^n}{(n-1)!} f^{(n)}(a + \theta h)\int_0^1 (1-t)^n\,\mathrm{d}t \\ &= \frac{h^n}{n!} f^{(n)}(a + \theta h). \end{aligned}$$

Proof of TT: Cauchy Remainder 3.23. Take $g = 1$.

$$\begin{aligned} R_{n,f,a}(h) &= \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^n(a + \theta h)\int_0^1 \mathrm{d}t \\ &= \frac{h^n}{n!}(1-\theta)^{n-1} f^{(n)}(a + \theta h). \end{aligned}$$

4.5 Improper Integrals

In this section, we will integrate functions of unbounded domain and unbounded image.

Definition 4.29 (Improper Integrals: Unbounded Domain)

Suppose $f : [0, \infty) \to \mathbb{R}$ is integrable on $[a, R]$ for every $a < R < \infty$, and set

$$\begin{aligned} F : [a, \infty) &\to \mathbb{R} \\ R &\mapsto \int_a^R f(x)\,\mathrm{d}x. \end{aligned}$$

Then we say that $\int_a^\infty f(x)\,\mathrm{d}x$ exists (converges) if

$$\lim_{R\to\infty} = L \in \mathbb{R}$$

then we set $\int_a^\infty f(x)\,\mathrm{d}x = L$. Otherwise, we say that $\int_a^\infty f(x)\,\mathrm{d}x$ does not exist.

If $f : \mathbb{R} \to \mathbb{R}$ is such that $\int_a^\infty f(x)\,\mathrm{d}x = L_1$ and $\int_{-\infty}^a f(x)\,\mathrm{d}x = L_2$, then we say $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ exists, and set

$$\begin{aligned}\int_{-\infty}^\infty f(x)\,\mathrm{d}x &= L_1 + L_2 \\ &= \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x \\ &= \lim_{R\to\infty} \int_{-R}^a f(x)\,\mathrm{d}x + \lim_{r\to\infty} \int_a^r f(x)\,\mathrm{d}x.\end{aligned}$$

> ***Remark.*** This is different from
>
> $$\lim_{R\to\infty} \left[\int_{-R}^R f(x)\,\mathrm{d}x\right].$$
>
> See further discussion in Example Sheet 4.

Example 4.30

- $\int_1^\infty x^p\,\mathrm{d}x = \lim_{R\to\infty} \int_1^R \frac{1}{x^p}\,\mathrm{d}x$ exists iff $x \geqslant 1$.
- $\int_2^\infty \frac{1}{x\log^2 x}\,\mathrm{d}x = \lim_{R\to\infty} \int_x^R \frac{\mathrm{d}x}{x\log^2 x} = \frac{1}{\log 2}$.
- $\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x$ exists assuming knowledge of normal distribution.

Lecture 20 · 2026-03-10

We need some convergence tests to have a proper proof for the last example.

Proposition 4.31 (Comparison Test for Integrals)

If $f, g : [a, \infty) \to \mathbb{R}$ satisfy $0 \leqslant f(x) \leqslant g(x)$ for all $x \geqslant a$, then

1. $\int_a^\infty g(x)\,\mathrm{d}x$ converges implies $\int_a^\infty f(x)\,\mathrm{d}x$ converges, and $\int_a^\infty f(x)\,\mathrm{d}x \leqslant \int_a^\infty g(x)\,\mathrm{d}x$.
2. $\int_a^\infty f(x)\,\mathrm{d}x$ diverges (to $+\infty$) implies $\int_a^\infty g(x)\,\mathrm{d}x$ diverges to $+\infty$.

Proof.

1. Let $F(R) = \int_0^R f(x)\,\mathrm{d}x$, note that this is increasing since $f \geqslant 0$. It is also bounded, since

   $$0 \leqslant F(R) \leqslant \int_0^R g(x)\,\mathrm{d}x \leqslant \int_0^\infty g(x)\,\mathrm{d}x < \infty.$$

   Since F is monotone and bounded, let $L = \sup_{R\geqslant a} F(R)$ exists. We claim that

   $$L = \lim_{R\to\infty} \int_0^R f(x)\,\mathrm{d}x.$$

   Indeed, by the definition of supremum, $\forall \varepsilon > 0, \exists R_0 \in [a, \infty)$ such that for $R \geqslant R_0$,

$$L-\varepsilon \leqslant F(R_0) \leqslant F(R) \leqslant L.$$

Hence,

$$L-\varepsilon \leqslant F(R) \leqslant L.$$

Taking limits to infinity,

$$L-\varepsilon \leqslant F(R) \leqslant L \quad \forall R \geqslant R_0 \quad \Rightarrow \lim_{R\to\infty} F(R) = L.$$

2. Since $f \geqslant 0$, $\lim_{R\to\infty} \underbrace{\int_0^R f(x)\,\mathrm{d}x}_{F(R)} = +\infty$ necessarily. Hence $\forall L > 0$, $\exists R$ such that $\forall r \geqslant R, F(r) > L$.

   But

$$\int_0^R g(x)\,\mathrm{d}x \geqslant F(R) > L.$$

   Hence $\int_0^\infty g(x)\,\mathrm{d}x$ diverges to $+\infty$.

Example 4.32

If $x \geqslant 1$, $x^2 \geqslant x$. Hence $\mathrm{e}^{-x^2} \leqslant \mathrm{e}^{-x}$ using properties of exponentials. Thus,

$$\int_1^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x \leqslant \int_1^\infty \mathrm{e}^{-x}\,\mathrm{d}x = \left[-\frac{1}{\mathrm{e}^x}\right]_1^\infty = \frac{1}{\mathrm{e}}$$

$$\Rightarrow \int_{-\infty}^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = 2\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = 2\underbrace{\int_1^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x}_{\frac{2}{\mathrm{e}}} + 2\underbrace{\int_0^1 \mathrm{e}^{-x^2}\,\mathrm{d}x}_{\text{finite since } \mathrm{e}^{-x^2} \text{ is integrable}}.$$

Indeed, $\int_{-\infty}^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x$ converges.

Proposition 4.33 (Ratio Test for Integrals)

Let $f, g : [0,\infty) \to \mathbb{R}$ satisfy $f, g \geqslant 0$, and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \in (0,\infty),$$

then

$$\int_a^\infty f(x)\,\mathrm{d}x \text{ converges} \Leftrightarrow \int_a^\infty g(x)\,\mathrm{d}x \text{ converges.}$$

Proof.

> ***Exercise.*** This is an application of the comparison test. Proof left as an exercise.

Remark. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then $f(x) \leqslant g(x)$ for very large x, hence by Comparison Test for Integrals 4.31 from the large x onwards,

$$\int_a^\infty g(x)\,\mathrm{d}x \text{ converges} \Rightarrow \int_a^\infty f(x)\,\mathrm{d}x \text{ converges.}$$

If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \infty$, then $f(x) \geqslant g(x)$ for very large x, and similarly by Comparison Test for Integrals 4.31 from the large x onwards,

$$\int_a^\infty g(x)\,\mathrm{d}x \text{ diverges} \Rightarrow \int_a^\infty f(x)\,\mathrm{d}x \text{ diverges.}$$

Example 4.34

- For e^{-x^2}, we have $\lim_{x\to\infty}\frac{e^{-x^2}}{e^{-x}} = 0$ and $\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = (1) < \infty$. By the remark,
  $$\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x < \infty.$$
- For $\int_1^\infty \frac{x}{x^4+1}\,\mathrm{d}x$ converges since
  $$\lim_{x\to\infty}\frac{\frac{x}{x^4+1}}{x^3} = 1 \in (0,\infty)$$
  and by $\int_1^\infty \frac{1}{x^3}\,\mathrm{d}x < \infty$ we have $\int_1^\infty \frac{x}{x^4+1}\,\mathrm{d}x < \infty$.

Exercise. On Example Sheet 4, there are examples of root test, Dirichlet test for integrals. Is a version of the n-th term test for integrals true or false?

We shall now consider improper integrals with bounded domain but isolated singularity.

Definition 4.35 (Improper Integrals: Isolated Singularity)

Let $f : (a,b] \to \mathbb{R}$ be integrable on $[a+\delta, b]$ for any $0 < \delta \leqslant b-a$. Set

$$\begin{aligned} F : (0, b-a] &\to \mathbb{R} \\ \delta &\mapsto \int_{a+\delta}^b f(x)\,\mathrm{d}x. \end{aligned}$$

Then, we say $\int_a^b f(x)\,\mathrm{d}x$ exists (converges) if $\lim_{\delta\to 0} f(\delta)$ exists and is finite, and we say

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\delta\to 0} F(\delta).$$

Otherwise, we say it does not exist (converge).

If $f : [B,b] \setminus \{a\}$ is such that $\int_B^a f(x)\,\mathrm{d}x = \ell_1 \in \mathbb{R}$ and $\int_a^b f(x)\,\mathrm{d}x = \ell_2 \in \mathbb{R}$, then say

$$\int_B^b f(x)\,\mathrm{d}x \text{ exists,} \quad \text{and} \quad \int_B^b f(x) = \ell_1 + \ell_2.$$

Remark.

$$\begin{aligned}\int_B^b f(x)\,\mathrm{d}x &= \int_B^a f(x)\,\mathrm{d}x + \int_a^b \mathrm{d}(x)\,\mathrm{d}x \\ &= \lim_{\delta\to 0}\int_B^{a-\delta} f(x)\,\mathrm{d}x + \lim_{\sigma\to 0}\int_{a+\sigma}^b f(x)\,\mathrm{d}x\end{aligned}$$

In general, this is not equal to

$$\lim_{\delta\to 0}\left[\int_B^{a-\delta} f(x)\,\mathrm{d}x + \int_{a+\delta}^b f(x)\,\mathrm{d}x\right].$$

This can be seen by taking $f(x) = \frac{1}{x}$.

Example 4.36

- $\int_0^1 \frac{1}{x^p}\,\mathrm{d}x$ converges iff $p < 1$, since as $\delta \to 0$

$$\int_\delta^1 \frac{1}{x^p}\,\mathrm{d}p = \begin{cases} p = 1: & [\log_x]_\delta^1 = -\log\delta \to -\infty \\ p \neq 1: & \left[\frac{x^{1-p}}{1-p}\right]_\delta^1 = \frac{\delta^{1-p}}{p-1} \to 0 \text{ iff } p < 1. \end{cases}$$

- Consider

$$\int_0^{\frac{1}{2}} \frac{\mathrm{d}x}{x\log^p(x)} = \int_\infty^{\log 2}\left(-\frac{1}{u^p}\right)\mathrm{d}u = \int_{\log 2}^\infty \frac{1}{u^p}\,\mathrm{d}u$$

  which converges iff $p > 1$.

Actually, we can always reduce, by substitution, an improper integral of function with isolated singularities to improper integrals with unbounded domain.

Lemma 4.37

Let $f : (a, b] \to \mathbb{R}$ be Riemann integrable on $[a+\delta, b]$ for all $\delta \in (0, b-a)$. Choose $\varphi : [c, \infty] \to [a, b]$ on C^1 strictly decreasing bijection with $\varphi(c) = b$, $\lim_{t\to\infty}\varphi(t) = a$. Then

$$\int_a^b f(x)\,\mathrm{d}x \text{ exists} \Leftrightarrow \int_c^\infty f(\varphi(t))(-\varphi'(t))\,\mathrm{d}t \text{ exists.}$$

Lecture 21 · 2026-03-12

5 Sequences and Series of Functions

We now have sufficient tools to revisit and generalise notions in Section 1.

5.1 Introduction

Definition 5.1 (Sequence of Functions)

A **sequence** of complex-valued functions $(f_n)_{n\in\mathbb{N}}$ on a set X is an enumerated list $(f_1, f_2, \ldots)$ where each element is a function $f_i : X \to \mathbb{C}$.

If, for each $x \in Y \subseteq X$, the numerical sequences $(f_n(x))_{n\in\mathbb{N}}$ converges, we can define a function $f : Y \subseteq X \to C$ such that

$$x \mapsto f(x) = \lim_{n\to\infty} f_n(x)$$

Definition 5.2 (Series of Functions)

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued functions. We call the enumerated sum

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + \ldots$$

a **series** of complex-valued functions on X.

If, for each $x \in Y \subseteq X$, the numerical series $\sum_{n=1}^{\infty} f_n(x)$ converges, we can define a function $f : Y \subseteq X \to C$ such that

$$x \mapsto f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Since we have discussed continuity, differentiability and integrability of functions, it is natural to ask whether these properties are preserved under limits.

For the easiest case of continuity we are asking whether

$$\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{n\to\infty} f_n(a) = f(a) \overset{?}{=} \lim_{x\to a} f(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x).$$

Example 5.3

Swapping limits is not trivial. Consider $s_{n,m} = \frac{m}{m+n}$.

- Fixing n, $\lim_{m\to\infty} s_{n,m} = 1$ and hence $\lim_{n\to\infty} \lim_{m\to\infty} s_{n,m} = 1$.
- Fixing m, $\lim_{n\to\infty} s_{n,m} = 0$ and hence $\lim_{m\to\infty} \lim_{n\to\infty} s_{n,m} = 0$.

Remark. For a general and formal treatment on this issue, see Part IB Analysis II.

5.2 Basics on Power Series

Definition 5.4 (Power Series)

For $a \in \mathbb{C}$ and $(c_n)_{n\in\mathbb{Z}_{\geqslant 0}}$, we say

$$\sum_{n=0}^{\infty} \underbrace{c_n(x-a)^n}_{f_n(x)}$$

is a power series with centre a and coefficients (c_n).

Example 5.5

The Taylor series of a smooth function f around a is a power series, where

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Tautologically, any power series converges at its centre. We would like to consider if

$$\left\{x \in \mathbb{C} : \sum_{n=0}^{\infty} c_n(x-a)^n\right\}$$

has any points other than a. [Otherwise we can define a function by $\sum c_n(x-a)^n$ only at a, which is not very interesting.]

Lemma 5.6

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for some x and $|y-a| < |x-a|$, then $\sum_{n=0}^{\infty} c_n(y-a)^n$ converges absolutely.

Proof. Fix x such that $\sum c_n(x-a)^n$ converges, and let y be such that $|y-a| < |x-a|$. Then since $\sum c_n(x-a)^n$ converges, by n-th term test 1.27 we have $\lim_{n\to\infty} c_n(x-a)^n = 0$.

Hence $(c_n(x-a)^n)_{n\in\mathbb{N}}$ is bounded, so there exists $M > 0$ such that

$$\left|c_n(x-a)^n\right| \leqslant M \quad \forall n \in \mathbb{N}.$$

Thus

$$\left|c_n(y-a)^n\right| \leqslant \left|c_n(x-a)^n\right| \cdot \left|\frac{y-a}{x-a}\right|^n \leqslant Mr^n$$

where $r = \left|\frac{y-a}{x-a}\right| < 1$. Since $\sum Mr^n$ converges, by comparison test $\sum \left|c_n(y-a)^n\right|$ converges.

Definition 5.7 (Radius of Convergence)

Let $\sum c_n(x-a)^n$ be a power series. Then $R \in [0,\infty]$ is called the **radius of convergence** of the power series, if

- $\sum c_n(x-a)^n$ converges absolutely for all x such that $|x-a| < R$.
- if $R < \infty$, then $\sum c_n(x-a)^n$ diverges for all x such that $|x-a| > R$.

Remark. On $|x-a| = R$ [either 2 points if over $\mathbb{R}$, or a circle if over $\mathbb{C}$], this definition does not enforce anything.

Moreover, if $R = 0$, the power series only converges at a; if $R = \infty$, the power series converges absolutely for all $x \in \mathbb{C}$.

Proposition 5.8

Every power series has a radius of convergence.

Proof. Define

$$A := \left\{ r \geqslant 0 : \exists x \in \mathbb{C} \text{ with } |x-a| = r \text{ such that } \sum c_n(x-a)^n \text{ converges} \right\}.$$

Clearly $0 \in A$ and A is hence non-empty.

If A is unbounded, we set $R = \infty$; by Lemma 5.6, not only $\forall r \geqslant 0$, there exists x with $|x-a| = r$ such that $\sum c_n(x-a)^n$ converges, but in fact $\sum c_n(x-a)^n$ converges absolutely for all $x \in \mathbb{C}$.

Otherwise, if A is bounded, let $R = \sup A$. Then

- $\sum c_n(x-a)^n$ diverges for $|x-a| > R$ by the definition of A
- if $|y-a| < R = \sup A$, then $\exists r \in A$ with $|y-a| < r \leqslant R$. Hence $\exists x \in \mathbb{C}$ with $|x-a| = r > |y-a|$, such that $\sum c_n(x-a)^n$ converges. By Lemma 5.6, $\sum c_n(y-a)^n$ converges absolutely.

To compute the radius of convergence, we can use our usual tests for series.

Proposition 5.9 (Root Test for Power Series)

Let (c_n) be a sequence in $\mathbb{C}$ such that

$$L = \lim_{n\to\infty} \sqrt[n]{|c_n|}$$

exists. Then the power series $\sum c_n(x-a)^n$ has radius of convergence

$$R = \frac{1}{L}$$

with the convention that $R = 0$ if $L = \infty$ and $R = \infty$ if $L = 0$.

Proposition 5.10 (Ratio Test for Power Series)

Let (c_n) be a sequence in $\mathbb{C}$ such that

$$L = \lim_{n\to\infty}\left|\left(\frac{c_{n+1}}{c_n}\right)\right|$$

exists. Then the power series $\sum c_n(x-a)^n$ has radius of convergence

$$R = \frac{1}{L}$$

with the convention that $R = 0$ if $L = \infty$ and $R = \infty$ if $L = 0$.

Exercise. Show that

- $\left|\frac{c_{n+1}}{c_n}\right| \to \infty$ implies $R = 0$.
- $\sqrt[n]{|c_n|} \to \infty$ implies $R = 0$.

Example 5.11

- $\sum \frac{x^n}{n!}$ converges on all of $\mathbb{C}$, since

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{n!}{(n+1)!}\right| = \frac{1}{n+1} \to 0 \Rightarrow R = \infty$$

- $\sum n!x^n$ converges only for $x = 0$.

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(n+1)!}{n!}\right| = n+1 \to \infty \Rightarrow R = 0$$

- $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ has $R = 1$ and converges absolutely at $|x| = 1$.

$$\left|\frac{c_{n+1}}{c_n}\right| = \left(\frac{n}{n+1}\right)^2 \to 1 \Rightarrow R = 1.$$

  At $|x| = 1$, we have $\sum_{n=1}^{\infty}\left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$.

- $\sum_{n=1}^{\infty}\frac{x^n}{n}$ has $R = 1$, but behavior at $|x| = 1$ is more subtle.

  If $x = 1$, then it diverges as harmonic series.

  If $x \neq 1$, then

$$\begin{aligned}(1-x)\sum_{n=1}^{N}\frac{x^n}{n} &= \sum_{n=1}^{n}\left(\frac{x^n}{n} - \frac{x^{n+1}}{n}\right)\\ &= \sum_{n=1}^{n}\left(\frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{n}\right) + x - \frac{x^{N+1}}{N+1}\\ &= -x\sum_{n=1}^{n}\frac{x^n}{n(n+1)} + x - \frac{x^{N+1}}{N+1}.\end{aligned}$$

  Hence

$$\sum_{n=1}^{N} \frac{x^n}{n} = \frac{x}{1-x} - \frac{x}{1-x} \sum_{n=1}^{n} \frac{x^n}{n(n+1)} + \frac{x^{N+1}}{(N+1)(1-x)}.$$

By taking the limit as $N \to \infty$, observe that $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$ converges absolutely, so $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges absolutely for all x with $|x| = 1$ and $x \neq 1$.

Lecture 22 · 2026-03-14

Remark. The set

$$\left\{x \in \mathbb{C} : \sum c_n (x-a)^n \text{ converges}\right\}$$

can be bigger or equal to

$$\{x \in \mathbb{C} : |x-a| < R\}.$$

5.3 Term-by-Term Operations on Power Series

Since $f_n(x) = c_n(x-a)^n$ are polynomials, they are hence continuous, integrable and differentiable. It is natural to consider whether possible to conclude something about the continuity, integrability, differentiability of

$$\begin{aligned} f : Y &\to \mathbb{C} \\ x &\mapsto \sum_{n=0}^{\infty} c_n (x-a)^n. \end{aligned}$$

And if so, whether

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \overset{?}{=} \sum_{n=0}^{\infty} c_n \frac{\mathrm{d}}{\mathrm{d}x} (x-a)^k$$

$$\int f(x)\,\mathrm{d}x \overset{?}{=} \sum_{n=0}^{\infty} c_n \int (x-a)^n\,\mathrm{d}x$$

Example 5.12

- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges on all of $\mathbb{C}$.

  $$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}}{\mathrm{d}x} (x^n) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

  which again converges on all of $\mathbb{C}$. This seems to be consistent.
- $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges on $D = \{x \in \mathbb{C} : |x| \leqslant 1\}$.

  $$\sum_{n=0}^{\infty} \frac{1}{n^2} \frac{\mathrm{d}}{\mathrm{d}x} (x^n) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

  which does not converge on all of D, but it converge on $\{x \in \mathbb{C} : |x| < 1\}$. This hints that the argument above is not totally correct.

We need to be careful that term-by-term operations will not hold on the entire set of convergence of power series, but we will show that they do within the radius of convergence.

Proposition 5.13 (Continuity of Power Series)

Let

$$\begin{aligned} B_R(a) &= \{x \in \mathbb{C} : |x-a| < R\} \\ D_r(a) &= \{x \in \mathbb{C} : |x-a| \leqslant r\}. \end{aligned}$$

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ have a radius of convergence $R > 0$. Then

$$\begin{aligned} f : B_R(a) &\to \mathbb{C} \\ x &\mapsto \sum_{n=0}^{\infty} c_n(x-n)^n \end{aligned}$$

is continuous inside $D_r(a)$ for every $r < R$.

Proof. [Non-examinable.]

Let

$$\begin{aligned} S_N(x) &= \sum_{n\leqslant N} c_n(x-a)^n, &&\text{which is a polynomial of degree } N \\ T_N(x) &= \sum_{n\geqslant N+1} c_n(x-a)^n, &&\text{which is a "tail"} \end{aligned}$$

and note the tail estimate

$$|T_N(x)| \leqslant \sum_{n\geqslant N+1} |c_n||x-a|^n \leqslant \sum_{n\geqslant N+1} |c_n|r^n \quad \forall x \in D_r(a).$$

By the definition of radius of convergence, $\forall \varepsilon > 0, \exists N = N(\varepsilon)$ such that

$$\sum_{n\geqslant N+1} |c_n|r^n < \varepsilon \Rightarrow |T_N(x)| < \varepsilon \quad \forall x \in D_r(a).$$

Take $x_0 \in D_r(a)$. We want to show that it is continuous at x_0. We have

$$\begin{aligned} |f(x) - f(x_0)| &= |S_N(x) - S_N(x_0) + T_N(x) - T_N(x_0)| \\ &\leqslant |S_N(x) - S_N(x_0)| + |T_N(x)| + |T_N(x_0)| \\ &< |S_N(x) - S_N(x_0)| + 2\varepsilon \quad \forall x \in D_r(a). \end{aligned}$$

To conclude the proof, choose $\delta = \delta(\varepsilon)$ so that

$$|S_N(x) - S_N(x_0)| < \varepsilon$$

which is possible because S_N is a polynomial.

Proposition 5.14 (Integration of Power Series)

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ have a radius of convergence $R > 0$. Then

$$\begin{aligned} f : (a-R, a+R) &\to \mathbb{R} \\ x &\mapsto \sum_{n=0}^{\infty} c_n (x-a)^n \end{aligned}$$

is integrable on $[a-r, a+r]$ for every $r < R$, and

$$\begin{aligned} \int_a^x f(t)\,\mathrm{d}t &= \sum_{n=0}^{\infty} c_n \int_a^x (t-a)^n\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}. \end{aligned}$$

Proof. [Non-examinable.]

We use the same split

$$f(x) = S_N(x) + T_N(x).$$

Then

$$\begin{aligned} \left| \int_a^x f(t)\,\mathrm{d}t - \sum_{n \leqslant N} \frac{c_n(x-a)^{n+1}}{n+1} \right| &= \left| \int_a^x f(t)\,\mathrm{d}t - \int_a^x S_N(t)\,\mathrm{d}t \right| \\ &= \left| \int_a^x T_N(t)\,\mathrm{d}t \right| \\ &\leqslant \sup_{|t-a|<r} |T_N(t)| \left| \int_a^x \mathrm{d}t \right| \\ &\leqslant r \cdot \sup_{|t-a|<r} |T_N(t)| \\ &< r\varepsilon \end{aligned}$$

if we choose N large enough. Hence

$$\int_a^x f(t)\,\mathrm{d}t = \lim_{N\to\infty} \sum_{n \leqslant N} \frac{c_n(x-a)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}.$$

Proposition 5.15 (Differentiation of Power Series)

Let $\sum c_n(x-a)^n$ and f as in the previous proposition. Then f is differentiable on $(a-R, a+R)$, and

$$f'(x) = \sum_{n=0}^{\infty} c_n \frac{\mathrm{d}}{\mathrm{d}x}(x-a)^n = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}.$$

Proof. [Non-examinable.]

If $g(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ has a radius of convergence $R' \geqslant R$, then g is continuous on $[a-r, a+r]$ for every $r < R$ by Proposition 5.13, and by Proposition 5.14 we have

$$\int_a^x g(t)\,\mathrm{d}t = \sum_{n=1}^{\infty} nc_n \int_a^x (t-a)^{n-1}\,\mathrm{d}t = \sum_{n=1}^{\infty} c_n(x-a)^n = f(x) - c_0.$$

Hence, by Fundamental Theorem of Calculus 4.21, we have

$$\begin{aligned} f'(x) &= g(x) \quad \forall x \in [a-r, a+r], \forall r < R \\ \Rightarrow f'(x) &= g(x) \qquad\quad \forall x \in (a-R, a+R) \end{aligned}$$

It remains to show that new series has radius of convergence $\geqslant R$. Take $r < R$, then we have some $s \in (r, R)$. THen

$$\begin{aligned} \left|nc_n(x-a)^{n-1}\right| &\leqslant |c_n| s^n \frac{n}{s}\left(\frac{|x-a|}{s}\right)^{n-1} \\ &\leqslant |c_n| s^n \frac{n}{s}\left(\frac{r}{s}\right)^{n-1} \qquad \forall |x-a| < r. \end{aligned}$$

Since $s < R$, $|c_n| s^n \xrightarrow[n\to\infty]{} 0$ by the definition of radius of convergence. Hence $\exists M$ such that $|c_n| s^n < \frac{M}{s}$ for all $n \geqslant 0$. Thus

$$\left|nc_n(x-a)^{n-1}\right| \leqslant M\frac{n}{s}\left(\frac{r}{s}\right)^{n-1} \quad \forall |x-a| < r.$$

Now

$$\sum_{n=0}^{\infty} \frac{Mn}{s}\left(\frac{r}{s}\right)^{n-1} \text{ converges since } \frac{r}{s} < 1.$$

Hence,

$$\sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \text{ converges absolutely for all } |x-a| < r, \forall r < R.$$

Example 5.16

- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has $R = \infty$. So

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} \frac{x^n}{n!} &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \int_0^x \sum_{n=0}^{\infty} \frac{x^n}{n!}\,\mathrm{d}t &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)(n!)} = \sum_{n=1}^{\infty} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1. \end{aligned}$$

- $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ has $R = 1$. For $|x| < 1$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=1}^{\infty} \frac{x^n}{n^2} &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \\ \int_0^x \sum_{n=1}^{\infty} \frac{x^n}{n^2}\,\mathrm{d}t &= \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)(n^2)}. \end{aligned}$$

Lecture 23 · 2026-03-17

5.4 Exponential and Logarithms

Exercise. Using familiar properties of exp, namely $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$, to show that they Taylor series of e^x at $x = 0$ is, with radius of convergence $R = \infty$,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

More ambitiously, Taylor's theorem says that $\exists \xi \in [0, x]$

$$R_{N,0}(x) - \sum_{n \leqslant N-1} \frac{x^n}{n!} = \frac{\mathrm{e}^\xi}{N!} x^N.$$

For fixed $x \in \mathbb{R}$,

$$\left|R_{N,0}(x)\right| \leqslant \left|\mathrm{e}^\xi\right| \frac{|x|^N}{N!} \to 0 \quad \text{as } N \to \infty.$$

Hence $\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.

Lemma 5.17

We define

$$\begin{aligned} e : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \sum_{n=0}^{\infty} \frac{z^n}{n!} \end{aligned}$$

with radius of convergence $R = \infty$.

With this definition,

1. e is smooth with $e'(z) = e(z)$,
2. $e(0) = 1$,
3. $e(a + b) = e(a)e(b)$.

Proof. (1) and (2) are immediate from what we know about power series. For (3), let

$$f(z) = e(a + b - z)e(z).$$

Then

$$\begin{aligned} f'(z) &= -e'(a + b - z)e(z) + e(a + b - z)e'(z) \\ &= -f(z) + f(z) = 0 \quad \forall z. \end{aligned}$$

Hence f is constant on $\mathbb{C}$, and thus

$$e(a)e(b) = f(a) = f(0) = e(a + b).$$

Lemma 5.18

Consider $e : \mathbb{R} \to \mathbb{R}$, the restirction of e to the real axis. Then

1. e is smooth with $e'(x) = e(x)$,
2. $e(x+y) = e(x)e(y)$,
3. $e(x) > 0$ for all $x \in \mathbb{R}$,
4. e is strictly increasing,
5. $e(x) \to \infty$ as $x \to \infty$, and $e(x) \to 0$ as $x \to -\infty$,
6. $e : \mathbb{R} \to (0, \infty)$ is a bijection.

Proof. (1) and (2) follows immediately from Lemma 5.17.

For $x > 0$, every term in the series is non-negative, so

$$e(x) > 1 + x \Rightarrow \begin{cases} e(x) > 0 & \text{for } x \geqslant 0 \\ e(x) \to \infty & \text{as } x \to \infty \end{cases}$$

For $x < 0$, $e(-x)e(x) = e(0) = 1$. Hence $e(-x) = \frac{e(0)}{e(x)} = \frac{1}{e(x)}$. Then

$$\begin{cases} e(x) > 0 & \forall x < 0 \\ e(-x) \to 0^+ \text{ as } x \to -\infty \end{cases}$$

It follows that

- $e'(x) = e(x) > 0$ gives that e is strictly increasing, and that e is injective.
- given $y \in (0, \infty) = (\lim_{x \to -\infty} e(x), \lim_{x \to \infty} e(x))$, there are $a, b \in \mathbb{R}$ such that $e(a) < y < e(b)$. By Intermediate Value Theorem 2.22, $\exists x \in [a, b]$ such that $e(x) = y$. Hence e is surjective.

Thus we have proved all statements.

Since $e : \mathbb{R} \to (0, \infty)$ is a bijection, it must have an inverse. We will call it $\ell : (0, \infty) \to \mathbb{R}$.

Lemma 5.19

For ℓ as defined above, we have

1. $\ell : (0, \infty) \to \mathbb{R}$ is a bijection with $\ell(e(x)) = x$ for all $x \in \mathbb{R}$, $e(\ell(y)) = y$ for all $y \in (0, \infty)$.
2. ℓ is smooth and monotone with $\ell'(y) = \frac{1}{y}$.
3. $\ell(1) = 0$ and $\ell(y) = \int_1^y \frac{1}{t} \, dt$.
4. $\ell(yz) = \ell(y) + \ell(z)$
5. $\ell(y) \to \infty$ as $y \to \infty$, and $\ell(y) \to -\infty$ by $y \to 0^+$.

Proof. (1) follows from the definition of inverse. It also gives us that $\exists u, v$ such that

$$\ell(yz) = \ell(e(u)e(v)) = \ell(e(u+v)) = u + v = \ell(y) + \ell(z)$$

where $y = e(u) \Leftrightarrow u = \ell(y)$, and $z = e(v) \Leftrightarrow v = \ell(z)$.

Using Inverse Function Theorem 3.13, we get that ℓ is smooth, and $\ell'(y) = \frac{1}{e'(\ell(y))} = \frac{1}{y}$.

Since $y > 0$ in the domain of ℓ, ℓ is monotone.

Since $e(0) = 1 \Leftrightarrow e(1) = 0$, by Fundamental Theorem of Calculus 4.21,

$$\begin{aligned}\ell(y) &= \int_1^y \ell'(t)\,\mathrm{d}t \\ &= \int_1^y \frac{1}{t}\,\mathrm{d}t.\end{aligned}$$

> ***Exercise.*** The final part is left as an exercise.

Remark.

$$\begin{aligned}\ell(1+y) = \int_1^y \frac{\mathrm{d}t}{1+t} &= \int_1^y \frac{1}{1-(-t)}\,\mathrm{d}t \\ &= \int_1^y \underbrace{\sum_{n=0}^{\infty} (-1)^n t^n}_{R=1}\,\mathrm{d}t \quad \text{given } |t| \leqslant |y| < 1 \\ &= \sum_{n=0}^{\infty} \int_1^y (-1)^n t^n\,\mathrm{d}t \qquad \forall |y| < 1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} y^{n+1} \qquad \forall |y| < 1.\end{aligned}$$

One can push this, in Part IB Analysis II, to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ell(2).$$

We shall define

$$\Gamma_\alpha(x) = e(\alpha \ell(x))$$

[The aim is to show that $\Gamma_\alpha(x) = x^\alpha$.]

Lemma 5.20

Let $x, y > 0$, $\alpha, \beta \in \mathbb{R}$. Then

1. $\Gamma_\alpha(xy) = \Gamma_\alpha(x)\Gamma_\alpha(y)$
2. $\Gamma_{\alpha+\beta}(x) = \Gamma_\alpha(x)\Gamma_\beta(x)$
3. $\Gamma_\alpha\big(\Gamma_\beta(x)\big) = \Gamma_{\alpha\beta}(x)$
4. $\Gamma_1(x) = x$, $\Gamma_0(x) = 1$.

Proof. (1) and (2) follow by group isomorphism properties of e and ℓ. (4) is clear from the analogous statements for e and ℓ. Now for (3),

$$\begin{aligned}\Gamma_\alpha\Big(\Gamma_\beta(x)\Big) &= e(\alpha\ell(e(\beta\ell(x)))) \\ &= e(\alpha\beta\ell(x)) = \Gamma_{\alpha\beta}(x).\end{aligned}$$

Corollary 5.21

Take $p, q \in \mathbb{Z}$, then

- $\Gamma_p(x) = \underbrace{\Gamma_1(x) \cdot \cdots \cdot \Gamma_1(x)}_{p \text{ products}} = x^p$,
- $\Gamma_{-p}(x) = x^{-p}$ by (3),
- $\Big(\Gamma_{\frac{1}{p}}(x)\Big)^p = \underbrace{\Gamma_{\frac{1}{p}}(x) \cdot \cdots \cdot \Gamma_{\frac{1}{p}}(x)}_{p \text{ products}} = \Gamma_1(x) = x$, and so $\Gamma_{\frac{1}{p}}(x) = x^{\frac{1}{p}}$,
- $x^{\frac{p}{q}} = \Gamma_{\frac{p}{q}}(x)$.

We have that $\Gamma_\alpha(x) = x^\alpha$ for $\alpha \in \mathbb{Q}$. Thus

$$e(x) = e(x \log \mathrm{e}) = \Gamma_x(\mathrm{e}) = \mathrm{e}^x.$$

This allows us to identify $e(x) = \mathrm{e}^x$ and use standard notation. We shall write $\ell = \log$ from now on.

Proposition 5.22 (Exponentials, Powers and Logarithms)

1. $x^r \mathrm{e}^{-x} \to 0$ as $x \to \infty$,
2. $x^{-r} \log x \to 0$ as $x \to \infty$,
3. $x^r \log x \to 0^+$ as $x \to 0^+$.

Proof.

1. Let $x > 1$,

$$\mathrm{e}^x = \sum_{k \geqslant 0} \frac{x^k}{k!} > \frac{x^n}{n!}$$

   for any $n \in \mathbb{N}, x > 0$. Choose $n - r \geqslant 1$. Then

$$0 \leqslant \frac{x^r}{\mathrm{e}^x} \leqslant \frac{n!}{x^{n-r}} \leqslant \frac{n!}{x} \to 0 \quad \text{as } x \to \infty.$$

2. $t^{-1} \leqslant t^{\varepsilon-1}$, $t \geqslant 1$ and any $\varepsilon > 0$. Pick $\varepsilon \in (0, r)$. Then

$$0 \leqslant x^{-r} \log x = x^{-r} \int_1^x \frac{\mathrm{d}t}{t} \leqslant x^{-r} \int t^{\varepsilon-1}\,\mathrm{d}t \leqslant \frac{x^{\varepsilon-r}}{\varepsilon} \xrightarrow[n\to\infty]{} 0$$

3. $\lim_x \to 0^+ x^r \log x \overset{x=\mathrm{e}^{-t}}{=} -\lim_{t\to\infty} t\mathrm{e}^{-rt} \overset{y=rt}{=} = -\lim_{y\to 0} \frac{y\mathrm{e}^{-y}}{r} = 0.$

Lecture 24 · 2026-03-19

5.5 Trigonometric Functions

With A-Level knowledge about sin and cos, we can calculate the Taylor series of them at $x = 0$. Moreover, using Taylor's theorem, we can show that for all $x \in \mathbb{R}$,

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$

Lemma 5.23

We define

$$\begin{aligned} c &: \mathbb{C} \to \mathbb{C} \\ &z \mapsto \frac{1}{2}[e(\mathrm{i}\, z) + e(-\mathrm{i}\, z)] = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}, \\ s &: \mathbb{C} \to \mathbb{C} \\ &z \mapsto \frac{1}{2\,\mathrm{i}}[e(\mathrm{i}\, z) - e(-\mathrm{i}\, z)] = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}. \end{aligned}$$

Note that both series have radius of convergence $R = \infty$. We will show that c and s are the complex cosine and sine functions, respectively.

With this definition, for $w, z \in \mathbb{C}$,

1. $c(0) = 1, s(0) = 0$
2. $c'(z) = -s(z), s'(z) = c(z)$
3. $s(z + w) = s(z)c(w) + c(z)s(w)$
4. $c(z + w) = c(z)c(w) - s(z)s(w)$
5. $c^2(z) + s^2(z) = 1.$

Proof. All statements follow from the definition and properties of e. For example, for (4),

$$\begin{aligned} c(z)c(w) - s(z)s(w) &= \frac{1}{4}\left(\mathrm{e}^{\mathrm{i}\,z} + \mathrm{e}^{-\mathrm{i}\,z}\right)\left(\mathrm{e}^{\mathrm{i}\,w} + \mathrm{e}^{-\mathrm{i}\,w}\right) + \frac{1}{4}\left(\mathrm{e}^{\mathrm{i}\,z} - \mathrm{e}^{-\mathrm{i}\,z}\right)\left(\mathrm{e}^{\mathrm{i}\,w} - \mathrm{e}^{-\mathrm{i}\,w}\right) \\ &= \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\,z}\mathrm{e}^{\mathrm{i}\,w} + \mathrm{e}^{-\mathrm{i}\,z}\mathrm{e}^{-\mathrm{i}\,w}\right) \\ &= c(z + w). \end{aligned}$$

Taking $z = -w$, we get (5).

Remark. From (5), $\forall x \in \mathbb{R}$,

$$s^2(x) + c^2(x) = 1 \Rightarrow |s(x)| \leqslant 1, |c(x)| \leqslant 1.$$

> Note that this is generally not true for complex z.

Proposition 5.24 (Periodicity of Trignometric Functions)

There is a smallest positive ω such that $c\left(\frac{\omega}{2}\right) = 0, s\left(\frac{\omega}{2}\right) = 1$, and

1. $s : \mathbb{R} \to \mathbb{R}, c : \mathbb{R} \to \mathbb{R}$ are periodic with period 2ω, *i.e.*
$$s(x + 2\omega) = s(x), \quad c(x + 2\omega) = c(x) \quad \forall x \in \mathbb{R}.$$
2. $s(x + \omega) = -s(x), c(x + \omega) = -c(x)$ for all $x \in \mathbb{R}$.
3. $s\left(x + \frac{\omega}{2}\right) = c(x), c\left(x + \frac{\omega}{2}\right) = -s(x)$ for all $x \in \mathbb{R}$.

Proof. Once we have existence of smallest $\omega > 0$ such that $c\left(\frac{\omega}{2}\right) = 0, s\left(\frac{\omega}{2}\right) = 1$, the rest of the statements follow from Lemma 5.23.

Let us find the smallest such ω. Starting from looking at $x \in (0, 2)$, we have

$$c'(x) = -s(x) = -\left[\underbrace{x - \frac{x^3}{3!}}_{>0} + \underbrace{\frac{x^5}{5!} - \frac{x^7}{7!}}_{>0} + \cdots\right] < 0.$$

Thus $c(x)$ is strictly decreasing on $(0, 2)$, and thus it has at most one root in $(0, 2)$. To see the existence of the root,

$$c\left(\sqrt{2}\right) = \underbrace{1 - \frac{\left(\sqrt{2}\right)^2}{2!}}_{>0} + \underbrace{\frac{\left(\sqrt{2}\right)^4}{4!} - \frac{\left(\sqrt{2}\right)^6}{6!}}_{>0} + \cdots > 0$$

$$c\left(\sqrt{3}\right) = \underbrace{1 - \frac{\left(\sqrt{3}\right)^2}{2!} + \frac{\left(\sqrt{3}\right)^4}{4!}}_{-1/8} - \underbrace{\frac{\left(\sqrt{3}\right)^6}{6!} + \frac{\left(\sqrt{3}\right)^8}{8!}}_{>0} - \underbrace{\frac{\left(\sqrt{3}\right)^{10}}{10!} + \frac{\left(\sqrt{3}\right)^{12}}{12!}}_{>0} - \cdots < 0.$$

By Intermediate Value Theorem 2.22, there is a root in $\frac{\omega}{2} = \left(\sqrt{2}, \sqrt{3}\right) \subseteq (0, 2)$. Now

$$s^2\left(\frac{\omega}{2}\right) = 1 - c^2\left(\frac{\omega}{2}\right) = 1 \Rightarrow s\left(\frac{\omega}{2}\right) = \pm 1.$$

But since $\frac{\omega}{2} \in (0, 2)$ we get $s\left(\frac{\omega}{2}\right) > 0$, so $s\left(\frac{\omega}{2}\right) = 1$.

Corollary 5.25

1. The function e^{ix} for $x \in \mathbb{R}$ is periodic with period 2ω
2. $e^{i\omega} = -1$ [*c.f.* Euler's identity $e^{i\pi} + 1 = 0$.]
3. $e^{\frac{i\omega}{2}} = i$.

Finally, we need to relate ω to the more familiar π.

Lemma 5.26

2ω is the perimeter of the unit circle,

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Proof. If we can show $\gamma : [0, 2\omega) \to \mathbb{S}^1$ with $t \mapsto e^{i\,t}$ is a bijection, then we get that

$$\text{perimeter of } \mathbb{S}^1 = \text{length of } \gamma = \int_0^{2\omega} |\gamma'(t)| \, dt = \int_0^{2\omega} \left| i\, e^{i\,t} \right| dt = 2\omega.$$

To show that γ is a surjection, take $z \in \mathbb{S}^1$ written as $z = a + i\, b$ with $a^2 + b^2 = 1$, and $a, b \in \mathbb{R}$. We have

- $c : [0, \omega] \to [-1, 1]$ is continuous and strictly monotone, hence $\exists t \in [0, \omega]$ with $c(t) = a$.
- $s : [0, \omega] \to [0, 1]$ is non0negative, hence $s(t) = \sqrt{1 - a^2}$. Then if $b = \sqrt{1 - a^2}$, we get $s(t) = b$. If $b = -\sqrt{1 - a^2}$, we get $b = -s(t) = s(2\omega - t)$.

Hence for every $z \in \mathbb{S}^1$, there is some $t \in [0, 2\omega)$ such that $\gamma(t) = z$.

To show that γ is an injection, since $e^{i\,t}$ has least positive positive 2ω, it is injective when restricted to $[0, 2\omega)$.

Hence $\omega = \pi$, and $c(z) = \cos z, s(z) = \sin z$ for all $z \in \mathbb{C}$. We can now define $\tan, \arccos$ etc.

We can further define, for $z \in \mathbb{C}$,

$$\cosh = \frac{1}{2}(e^z + e^{-z}) = \cos(i\, z), \quad \sinh = \frac{1}{2}(e^z - e^{-z}) = \infty \sin(i\, z).$$

Proposition 5.27

For $z \in \mathbb{C}$,

1. $\cosh'(z) = \sinh(z)$
2. $\sinh'(z) = \cosh(z)$
3. $\cosh^2(z) - \sinh^2(z) = 1$.

Exercise. The proof is left as an exercise. It should be similar to the proof of Lemma 5.23.

END OF DOCUMENT ■